UNIT I – TESTING OF HYPOTHESIS

Introduction

Population in statistics means the whole of the information which comes under the purview of statistical investigation. A part of the population selected for study is called a sample. When the sample is drawn properly, it is identical with its population almost in all respect.

Inferential statistics is used to measure behavior in samples to learn more about the behavior in populations that are very large or inaccessible. Samples are used because it is obvious how they are related to populations. For example, if we want to have an idea of the average income of the people of a country, we will have to collect all the earning individuals in the country-, which is quite difficult task. Hence samples are used.

Mean, median, mode, standard deviation are some examples of the statistical measure. It can be evaluated from the population and samples. A numerical measure of a sample is called a statistic. A numerical measure of a population is called a parameter. Population parameters are estimated by sample statistics. When a sample statistic is used to estimate a population parameters, the statistic is called an estimator of the parameter.

Define the terms: Statistic, Parameter.
MA 6452 MAY 18

Briefly explain about 'estimation of parameters'.
MA 8452 APR 22

Sampling Theory

The process of selecting a sample from an population/universe is called sampling. Theory of sampling is a study of relationship existing between a population and samples drawn from the population. The aim is to get information about the population by examining a sample of it.

A random sample is on in which each element of the population has an equal chance of inclusion in the sample. i.e. each part of the population has some pre assigned probability of being selected in the sample.

What is random sampling?.
MA 6452 NOV 15

Sampling Distribution

Sampling distribution of a statistic is the frequency distribution which is formed with various values of a statistic calculated from all samples of same size. i.e. For any sample $\{x_1, x_2, ..., x_n\}$ of a given finite population, we can compute statistics $t\{x_1, x_2, ..., x_n\}$ such as mean, variance etc. The set of all such statistics, one for each sample, is called the sampling distribution of a statistic.

What is sampling distribution?.
MA 8452 APR 22

Standard Error of a Statistic

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The standard deviation of the sampling distribution of a statistic is known as standard error. It is used to measure the variability of the values of a statistic.

Define the term Standard Error.
MA 6452 MAY 18

The Standard Errors of some of the well-known statistics, for large samples, are given below, where n is the sample size, σ^2 is the population variance.

Statistic	Standard Error
Sample Mean $\overline{x}$	$\frac{\sigma}{\sqrt{n}}$
Observed Sample Proportion p	$\sqrt{\frac{PQ}{n}}$
Sample Standard Deviation s	$\sqrt{\frac{\sigma^2}{2n}}$
Sample Variance s^2	$\sigma^2\sqrt{\frac{2}{n}}$

Example: A telephone tower monitored for an hour was found to have an estimated mean of 20 signals transmitted per minute. The variance is known to be 4. Find the standard error for mean.

Solution: Given $\sigma^2 = 4.$ Also $\overline{x} = 20/\min$ and $n = 1\ hr = 60\min$

$$\text{Standard Error } \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{60}} = 0.2582$$

Test of Significance

Sampling theory deals with a problem of testing hypothesis. A hypothesis is a statement about the population parameter, i.e. a conclusion tentatively drawn on logical basis.

The method in which we select samples to learn more about characteristics in a given population is called hypothesis testing. Hypothesis testing is a systematic way to test claims or population. i.e. It enables us to decide, on the basis of the results of the sample, whether (i) the deviation between the observed sample statistic and the hypothetical parameter value or (ii) the deviation between two samples statistics is significant

Procedure for testing a hypothesis

State the procedure followed in testing of hypothesis.
MA 6452 NOV 18

Step 1: Setting up of Null Hypothesis H_0: It is a definite statement about population parameter set up whether to accept or reject it. It states that there is no difference between the sample statistic and population parameter. To test the statement about population, hypothesis that it is true.

Step 2: Setting up Alternative Hypothesis H_1: It is a complementary statement to null hypothesis. It is set in such a way that the rejection of null hypothesis implies the acceptance of alternative hypothesis.

Step 3: Computation of test statistic:

For large sample $(n \geq 30)$, Z – statistic is used and it is defined as $Z = \dfrac{t - E(t)}{S.E.(t)} \sim N(0,1) \ as \ n \to \infty$

For small sample, the student's t–statistic is used and it is defined as $Z = \dfrac{Difference \ of \ means}{S.E.(\bar{x})}$ with $n-1$ degrees of freedom.

Types of Errors in Hypothesis Testing: There are two possible types of errors which may arise in testing a hypothesis.

Type I Error : Rejecting Null Hypothesis when it is true.

Step 4: The probability of making Type I error is denoted by α, the level of significance. The probability level below which we reject a null hypothesis is called the level of significance. In other words. level of significance is the size of the type I error. If level of significance in 5%, then we say that the probability for committing Type I error is 0.05. This means that a correct decision is made 95% confidently.

Type II Error : Accepting Null Hypothesis when it is wrong.

Define Type I error and Type II error in the sampling distribution. MA 6452 NOV 18

What is meant by level of significance?. MA 6452 NOV 17

Step 5: Critical or Rejection Region:

A region corresponding to a test statistic in the sample space which tends to rejection of H_0 is called critical region or region of rejection. The value of the test statistic is known as critical value z_α. The critical value separates the region of rejection from the acceptance region.

What is meant by critical region?. MA 6452 NOV 17

Define critical value of a test statistic. MA 3251 APR 22

Step 6:.Two tailed and One tailed test: The probability curve of a sampling distribution is a normal curve. The rejection may be represented by a area on each sides or by one side of the normal curve and the corresponding test is known as two tailed or one tailed respectively. In two tailed test, the alternative hypothesis is denoted by $\mu \neq \mu_0$. In one tailed test the same is denoted by $\mu < \mu_0$ or $\mu > \mu_0$.

Step 7: Conclusion: If $|Z| < Z_\alpha$, then we accept the null hypothesis. If $|Z| > Z_\alpha$, then we reject the null hypothesis.

Note:

1. Compare calculated value of z with the critical value z_α at level of significance α. The critical value of z_α of the test statistic for a two tailed test is given by $p\left(|z| > z_\alpha\right) = \alpha$. By symmetry of normal curve

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$$p(z > z_\alpha) + p(z < -z_\alpha) = \alpha$$
$$2p(z > z_\alpha) = \alpha$$
$$p(z > z_\alpha) = \frac{\alpha}{2}$$

In case of one tailed test, $p(z > z_\alpha) = \alpha$ if it is right tailed test; $p(z < -z_\alpha) = \alpha$ if it is left tailed.

2. The critical value of z for one tailed test at level of significance α is same as the critical value of z for two tailed test at level of significance 2α.

From the normal table, the critical values of z at different levels of significant are listed below:

	1%	5%	10%
Two tailed test	$\lvert z_\alpha \rvert = 2.58$	$\lvert z_\alpha \rvert = 1.96$	$\lvert z_\alpha \rvert = 1.645$
Right tailed test	$z_\alpha = 2.33$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left tailed test	$z_\alpha = -2.33$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

3. The values of the test statistic which separates the critical region and acceptance region is called critical values. This value is dependent on level of significance and alternative hypothesis.

Degrees of freedom

The number of independent variates used to compute the test statistic is known as the number of degrees of freedom. In general, the number of degrees of freedom is given by $v = n - k$, where n is the number of observations in the sample and k is the number of constraints imposed on them.

What do you mean by degree of freedom? MA 3251 APR 22

Test of significance for Single Proportion – Large Sample

Test of significance of the difference between sample proportion and population proportion **Working Rule**
When sample proportion p and population proportion P is given
• Set up null hypothesis $H_0: p = P$ (or) $H_0: P =$ Given Value • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test.

- Compute test statistic $z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$ where $Q = 1-P$, $n =$ sample size.
- Choose appropriate level of significance and table value of z_α (critical value)
- Compare calculated value of $|z|$ with the tabulated value.
- Conclusion: If $|z| < z_\alpha$ then accept H_0. If $|z| > z_\alpha$ then reject H_0.

Solved Problems

1. A coin is tossed 144 times and head appeared 80 times. Can we say that the coin is unbiased?

Probability of getting a head in a toss

$P = \frac{1}{2}$, *hence* $Q = 1-P = \frac{1}{2}$. Given $n = 144$

Given sample proportion $p = \frac{80}{144}$

Null hypothesis $H_0: P = \frac{1}{2}$ (the coin is unbiased)

Alternative hypothesis $H_1: P \neq \frac{1}{2}$ (the coin is biased)

Test statistic:

2. A coin is tossed 800 times and head appeared 350 times. Can we say that he has made a random tossing each time? (equivalently can we say that the coin is unbiased?

Probability of getting a head in a toss

$P = \frac{1}{2}$, *hence* $Q = 1-P = \frac{1}{2}$. Given $n = 800$

Given sample proportion $p = \frac{350}{800}$

Null hypothesis $H_0: P = \frac{1}{2}$ (random tossing is made)

Alternative hypothesis $H_1: P \neq \frac{1}{2}$ (coin is not randomly tossed

Test statistic:

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$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{80}{144} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2}\cdot\frac{1}{2}}{144}}}$$

$$= \frac{0.0555}{0.04166}$$

$$= 1.333$$

We choose 5% level of significance and hence the table value $z_{0.05} = 1.96$

Since calculated value of $|z| < z_\alpha$ then accept H_0.

i.e. the coin is unbiased

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$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{350}{800} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2}\cdot\frac{1}{2}}{800}}}$$

$$= \frac{-0.0625}{0.0176}$$

$$= -3.525$$

We choose 5% level of significance and hence the table value $z_{0.05} = 1.96$

Since calculated value of $|z| > z_\alpha$ then reject H_0.

i.e. the coin is not tossed randomly (equivalently the coin is not unbiased)

3. Experience has shown that 20% of a manufactured product is of top quality. In one day's production of 400 articles, only 50 are of top quality. Show that either the production of the day chosen was not a representative sample or the hypothesis of 20% was wrong.

Population probability for top quality product

$P = \frac{20}{100}$, *hence* $Q = 1 - P = \frac{80}{100}$. Given $n = 400$

Given sample proportion $p = \frac{50}{400}$

Null hypothesis $H_0 : P = \frac{1}{5}$ (20% products manufactured is of top quality)

4. In a city, a sample of 500 people, 280 are tea drinkers and the rest are coffee drinkers. Can we assume that both coffee and tea are equally popular in this city at 5% level of significance.

Population proportion for tea drinkers

$P = \frac{1}{2}$, *hence* $Q = 1 - P = \frac{1}{2}$. Given $n = 500$

Given sample proportion $p = \frac{280}{500}$

Null hypothesis $H_0 : P = \frac{1}{2}$ (tea and coffee are equally popular)

Alternative hypothesis $H_1 : P \neq \frac{1}{5}$ (20% products manufactured is not of top quality)

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{50}{400} - \frac{1}{5}}{\sqrt{\frac{\frac{1}{5} \cdot \frac{4}{5}}{400}}}$$

$$= \frac{-0.075}{0.02}$$

$$= -3.75$$

We choose 5% level of significance and hence the table value $z_{0.05} = 1.96$

Since calculated value of $|z| > z_\alpha$ then reject H_0.

i.e. 20% products manufactured is not of top quality.

Alternative hypothesis $H_1 : P \neq \frac{1}{2}$ (tea and coffee are not equally popular)

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{280}{500} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2} \cdot \frac{1}{2}}{500}}}$$

$$= \frac{0.06}{0.022}$$

$$= 2.68$$

We choose 5% level of significance and hence the table value $z_{0.05} = 1.96$

Since calculated value of $|z| > z_\alpha$ then reject H_0.

i.e. tea and coffee are not equally popular

5. In a city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

Population proportion of smokers in the city

$P = \frac{1}{2}$, *hence* $Q = 1 - P = \frac{1}{2}$. Given $n = 600$

Given sample proportion $p = \frac{325}{600}$

6. A manufacturer claimed that at least 95% of the items which he supplied confirmed to specifications. A sample of 200 pieces of items revealed that 18 were faulty. Test his claim at a significance level of 1%.

Population proportion of items supplied by him confirmed specification is

$P = \frac{95}{100}$, *hence* $Q = 1 - P = \frac{5}{100}$. Given $n = 200$

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Null hypothesis $H_0: P = \frac{1}{2}$ (smokers and non smokers are equal in the city)

Alternative hypothesis $H_1: P > \frac{1}{2}$ (right tailed test)

Test Statistic:

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{325}{600} - \frac{1}{2}}{\sqrt{\frac{\frac{1}{2}.\frac{1}{2}}{600}}}$$

$$= \frac{0.0417}{0.0204}$$

$$= 2.043$$

We choose 5% level of significance and hence the table value $z_{0.05} = 1.645$

Since calculated value of $|z| > z_\alpha$ then reject H_0.

i.e. majority of men in the city are smokers

Given sample proportion confirming to specification is $p = \frac{182}{200}$. (200–18, faulty items)

Null hypothesis $H_0: P = \frac{95}{100}$ (proportion of items confirming to the specification in the population 95%)

Alternative hypothesis $H_1: P < \frac{95}{100}$ (left tailed test)

$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$$

$$= \frac{\frac{5}{100} - \frac{95}{100}}{\sqrt{\frac{\frac{95}{100}.\frac{5}{100}}{200}}}$$

$$= \frac{-0.04}{0.0154}$$

$$= -2.6$$

We choose 5% level of significance and hence the table value $z_{0.01} = -2.33$

Since calculated value of $|z| > z_\alpha$ then reject H_0.

i.e. quantity of items confirming specification is less than 95%

Test of significance for Difference of Proportions – Large Samples

Working Rule
When $p_1, \; p_2$ be two sample proportions drawn from the same population or from two populations with the same proportion P is given
• Set up null hypothesis $H_0: P_1 = P_2$ (Population proportions are equal)

- Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test.
- Compute test statistic $z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$ where $Q = 1 - P$, $n_1, n_2 =$ sample sizes.

  If P is not known, then $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$
- Choose appropriate level of significance and table value of z_α (critical value)
- Compare calculated value of $|z|$ with the tabulated value.
- Conclusion: If $|z| < z_\alpha$ then accept H_0. If $|z| > z_\alpha$ then reject H_0.

Note: Suppose we want to test $H_0 : P_1 - P_2 = d_0$ against $H_1 : P_1 - P_2 > d_0$. Now $z = \frac{(p_1 - p_2) - d_0}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$.

Solved Problems

1. **In a large city A 20% of a random sample of 900 school boys had a slight physical defect. In another city B 18.5% of a random sample of 1600 school boys had the same effect. Is the difference between the proportions significant? (QC 53250 MA 6452 MAY 19)**

Null Hypothesis: $H_0 : P_1 = P_2$, the difference between the two proportions is not significant

Alternative Hypothesis: $H_1 : P_1 \neq P_2$

Given $p_1 = \frac{20}{100} = 0.2$ and $p_2 = \frac{18.5}{100} = 0.185$. Also $n_1 = 900$, $n_2 = 1600$

Hence $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{900(0.2) + 1600(0.185)}{900 + 1600} = 0.19$ and $Q = 1 - P = 0.81$

Therefore test statistic $z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_1}\right)}} = \frac{0.20 - 0.185}{\sqrt{(0.19)(0.81)\left(\frac{1}{900}+\frac{1}{1600}\right)}} = 0.918$

Table value of z at 5% level of significance is 1.96

Since calculated value of $|z| <$ tabulated value, we accept null hypothesis. Therefore the difference between the proportions are not significant.

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2. **400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test whether these two proportions are same.** **(QC 57506 MA 6452 MAY 16)**

Null Hypothesis: $H_0: P_1 = P_2$, the difference between the attitude of men and women as far as the proposal is concerned is not significant.

Alternative Hypothesis: $H_1: P_1 \neq P_2$

Given $p_1 = \frac{200}{400} = 0.5$ and $p_2 = \frac{325}{600} = 0.542$. Also $n_1 = 400, \quad n_2 = 600$

Hence $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{400(0.5) + 600(0.542)}{400 + 600} = 0.525$ and $Q = 1 - P = 0.475$

Therefore test statistic $z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_1}\right)}} = \frac{0.5 - 0.542}{\sqrt{(0.525)(0.475)\left(\frac{1}{400} + \frac{1}{600}\right)}} = \frac{-0.042}{\sqrt{0.032234}} = -1.302$

Table value of z at 5% level of significance is 1.96

Since calculated value of $|z| <$ tabulated value, we accept the null hypothesis. Therefore the difference between the attitude of men and women as far as the proposal is concerned is not significant.

3. **Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.**

(QC E3126 MA 2266 MAY 10)

Before increase of duty $p_1 = \frac{800}{1000} = 0.8$

After increase of duty $p_2 = \frac{800}{1200} = 0.666$. Also

$n_1 = 800, \quad n_2 = 1000$

4. **15.5% of a random sample of 1600 undergraduates were smokers whereas 20% of a random sample of 900 postgraduates were smokers in a state. Can we conclude that less number of undergraduates are smokers than the postgraduates?**

Smokers in undergraduate $p_1 = 15.5\% = 0.155$

Smokers in postgraduate $p_2 = 20\% = 0.2$.

Also $n_1 = 1600, \quad n_2 = 900$

Hence

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{1600(0.155) + 900(0.2)}{1600 + 900} = 0.1712$$

and $Q = 1 - P = 0.8288$

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Hence

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{800(0.8) + 1000(0.666)}{800 + 1000} = 0.7273$$

and $Q = 1 - P = 0.2727$

Null Hypothesis: $H_0 : P_1 = P_2$, the difference between the proportion of tea drinkers before and after imposing the excise duty is not significant.

Alternative Hypothesis: $H_1 : P_1 > P_2$ (right tailed test)

Therefore test statistic

$$z = \frac{(p_1 - p_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_1}\right)}}$$

$$= \frac{(0.8 - 0.67)}{\sqrt{(0.7273)(0.2727)\left(\frac{1}{1000} + \frac{1}{1200}\right)}}$$

$$= 6.81$$

Table value of z at 1% level of significance is 2.33

Since calculated value of $|z| >$ tabulated value, we reject the null hypothesis. Therefore there is significant decrease in the consumption of tea after the impose of duty.

Null Hypothesis: $H_0 : P_1 = P_2$, the difference between the proportion of smokers in UG and PG is not significant.

Alternative Hypothesis: $H_1 : P_1 < P_2$ (left tailed test)

Therefore test statistic

$$z = \frac{(p_1 - p_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_1}\right)}}$$

$$= \frac{(0.155 - 0.2)}{\sqrt{(0.1712)(0.8288)\left(\frac{1}{1600} + \frac{1}{900}\right)}}$$

$$= \frac{-0.045}{0.0156}$$

$$= -2.86$$

Table value of z at 5% level of significance is -1.645

Since calculated value of $|z| >$ modulus of tabulated value, we reject the null hypothesis. Therefore smoking habit is less among UG than among the PG.

5. A cigarette manufacturing firm claims that its brand A outsells its brand B by 8%. It is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B. Test whether the 8% difference is a valid claim.

Null Hypothesis: $H_0 : P_1 - P_2 = 0.08$, the difference between the sale of brand A and brand B is 8%.

Alternative Hypothesis: $H_1 : P_1 - P_2 \neq 0.08$

Proportion of preference of brand A $p_1 = \frac{42}{200} = 0.21$

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Proportion of preference of brand B $p_2 = \frac{18}{100} = 0.18$. Also $n_1 = 200, \quad n_2 = 100$

Hence $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{200(0.21) + 100(0.18)}{200 + 100} = 0.2$ and $Q = 1 - P = 0.8$

Therefore test statistic $z = \frac{(p_1 - p_2) - d_0}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_1}\right)}} = \frac{0.03 - 0.08}{\sqrt{(0.2)(0.8)\left(\frac{1}{200} + \frac{1}{100}\right)}} = \frac{-0.05}{0.0489} = -1.02$

Table value of z at 5% level of significance is 1.96

Since calculated value of $|z| <$ tabulated value, we accept the null hypothesis. Therefore the difference of 8% in the sale of brand A and brand B is a valid claim.

Chi Square Test for Population Variance – Small Sample

Test of significance for population variance **Working Rule**
When sample SD/Var s / s^2, size n and population SD/Var σ_0 / σ_0^2 is given
• Set up null hypothesis $H_0 : \sigma^2 = \sigma_0^2 \ \ or \ \ \sigma = \sigma_0$ • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test. • Compute test statistic $\chi^2 = \frac{n.s^2}{\sigma_0^2}$ where $\sigma_0 =$ Given population SD. • Choose appropriate level of significance and degrees of freedom $n-1$ and find table value of χ_α^2 (critical value) • Compare calculated value of $\lvert \chi^2 \rvert$ with the tabulated value. • Conclusion: If $\lvert \chi^2 \rvert < z_\alpha$ then accept H_0. Otherwise reject H_0.

Solved Problems

1. **It is believed that the precision (as measured by variance) of an instrument is no more than 0.16. Write down the null and alternative hypothesis for testing this belief. Carry out the**

test at 1% level of significance, given 11 measurements of the same subject on the instrument. 2.5, 2.3, 2.4, 2.3, 2.5, 2.7, 2.5, 2.6, 2.6, 2.7, 2.5 **(QC 72071 MA 6452 MAY 17)**

Let us evaluate sample mean $\bar{x}$ and sample S.D s:

x	2.5	2.3	2.4	2.3	2.5	2.7	2.5	2.6	2.6	2.7	2.5	**T: 27.6**
$(x-\bar{x})^2$	0	0.04	0.01	0.04	0	0.04	0	0.01	0.01	0.04	0	**T: 0.19**

$$\bar{x} = \frac{\sum x}{n} = \frac{27.6}{11} = 2.51 \quad \text{and} \quad s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{0.19}{10}} = 0.137$$

Null Hypothesis $H_0 : \sigma^2 = 0.16$ (the precision of the instrument is 0.16)

Alternative Hypothesis $H_1 : \sigma^2 > 0.16$

Test statistic $\chi^2 = \frac{n.s^2}{\sigma^2} = \frac{11(0.137)^2}{0.16} = 1.29$

Table value of χ^2 for $n-1=10$ degrees freedom at 5% level of significance is 23.2

Since calculated value of $\chi^2 <$ the tabulated value, we accept the null hypothesis.

i.e. the data are consistent with the hypothesis that the precision of the instrument is 0.16

2 **A random sample of size 25 from a population gives the sample SD 8.5. Test the hypothesis that the population SD is 10.**

Given $n = 25,\ s = 8.5\ \ and\ \ \sigma_0 = 10$

Null Hypothesis $H_0 : \sigma^2 = 100$ (the sample is taken from the population with SD=10)

Alternative Hypothesis $H_1 : \sigma^2 \neq 100$

Test statistic $\chi^2 = \frac{n.s^2}{\sigma^2} = \frac{25(8.5)^2}{100} = 18.06$

Table value of χ^2 for $n-1=24$ degrees freedom at 5% level of significance is 36.4

Since calculated value of $\chi^2 <$ the tabulated value, we accept the null hypothesis.

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i.e. the sample is taken from the population with SD=10

Chi Square Test for goodness of fit

There is always difference exists in the theoretical and experimental/observed values of an incident. The quantity χ^2 describes the magnitude of discrepancy between theory and observation.

1. What are the conditions for the validity of χ^2 test. **(QC 11395 MA 2266 MAY 11)**

To apply χ^2 test, total number of observations/frequencies should be at least 50. Also individual theoretical frequencies must be at least 10. Otherwise the smaller frequencies can be combined together before calculating (O–E). Also degree of freedom is calculated after re-grouping.

In general the degrees of freedom is $n-1$ but to fit Poisson distribution, it is $n-2$.

2. Write any two characteristics of χ^2 test. **(QC 53250 MA 6452 MAY 19)**

The shape of the distribution depends upon the number of degrees of freedom. If it is small, the curve is skewed to the right. If it is larger, the curve becomes more and more symmetrical.
The mean and variance of the χ^2 distribution are n and $2n$ respectively
Sum of independent χ^2 variates is also χ^2 variate.

3. State any two applications of Chi-square test. **(QC 50782 MA 6452 NOV 17)**

- It is used to determine whether an actual sample distribution matches a known theoretical distribution
- To test the independence of attributes
- To test if the hypothetical value of the population variance is $\sigma^2 = \sigma_0^2$ (say)

Working Rule
This test is used • **to fit theoretical distribution** • **to find the compatibility of observed and theoretical frequencies**
• Set up null hypothesis H_0 Observed and Theoretical frequencies are equal • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test. • Compute expected frequencies according to the question • Compute test statistic $\chi^2 = \sum_{i=1}^{n} \frac{(O-E)^2}{E}$.

- Choose appropriate level of significance and degrees of freedom $n-1$ and find table value of χ_{α}^{2} (critical value)
- Compare calculated value of $|\chi^2|$ with the tabulated value.
- Conclusion: If $|\chi^2| < z_{\alpha}$ then accept H_0. If $|\chi^2| > \chi_{\alpha}^{2}$ then reject H_0.

Solved Problems

1. Define Chi-square test for goodness of fit. **(QC E3126 MA 2266 APR 10)**

Chi-square test is used to find how good the theoretical distributions such as Binomial, Poisson etc. fit empirical distributions, distributions obtained from sample data.

2. The below table gives the number of aircraft accidents that occurred during the various days of a week. Test whether the accidents are uniformly distributed over the week.
(QC 20817 MA 8452 APR 22)

Day	:	Mon	Tue	Wed	Thu	Fri	Sat
No. of accidents	:	15	19	13	12	16	15

H_0 : Accidents are uniformly distributed over the week.
H_1 : Accidents are not uniformly distributed over the week.

Average accident per day = $\frac{15+19+13+12+16+15}{6} = \frac{90}{6} = 15$

Hence the frequency table is

X	0	1	2	3	4	5	Total
Observed Frequency – O	15	19	13	12	16	15	90
Expected Frequency – E	15	15	15	15	15	15	
$\frac{(O-E)^2}{E}$	0	1.066	0.266	0.6	0.066	0	1.998

Test statistic $\psi^2 = \sum \frac{(O-E)^2}{E} = 1.998$

Critical value of χ^2 at 5% level of significance for $\nu = 5$ degrees of freedom is 11.07

Since calculated value of $\chi^2 < \chi^2_{\alpha=0.05}$, accept the null hypothesis H_0.

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3. The number of automobile accidents in a certain locality was 12, 8, 20, 2, 14, 10, 15, 6, 9, 4. Are these frequencies in agreement with the belief that accident conditions were the same during this 10 week period. **(QC 27331 MA 6452 NOV 15)**

H_0 : Accidents are uniformly distributed over the 10 week period.

H_1 : Accidents are not uniformly distributed over the 10 week period.

Average accident per week = $\frac{12+8+20+2+14+10+15+6+9+4}{10} = \frac{100}{10} = 10$

Hence the frequency table is

X	1	2	3	4	5	6	7	8	9	10	Total
Observed Frequency O	12	8	20	2	14	10	15	6	9	4	
Expected Frequency E	10	10	10	10	10	10	10	10	10	10	
$\frac{(O-E)^2}{E}$	0.4	0.4	10	6.4	1.6	0	2.5	1.6	0.01	3.6	26.51

Test statistic $\chi^2 = \sum \frac{(O-E)^2}{E} = 26.51$

Critical value of χ^2 at 5% level of significance for $v = 10-1 = 9$ degrees of freedom is 16.9

Since calculated value of $\chi^2 > \chi^2_{\alpha=0.05}$, reject the null hypothesis H_0.
the frequencies does not support the belief that accident conditions were the same during this 10 week period.

4. Theory predicts the proportion of beans in the groups A, B, C, D as $9:3:3:1$. In an experiment among beans the numbers in the groups were 882, 313, 287 and 118. Does the experiment support the theory? **(QC 57506 MA 6452 MAY 16)**

H_0 : The theory fits well into the experiment. i.e. the experimental results support the theory.

H_1 : The experimental results does not support the theory.

Total number of beans = 1600. Divide this in the ratio 9 : 3 : 3 : 1. The expected frequencies are

$E(882) = \frac{89}{16} \times 1600 = 900$ $\qquad$ $E(313) = \frac{3}{16} \times 1600 = 300$

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$$E(287)=\frac{3}{16}\times 1600=300 \qquad E(118)=\frac{1}{16}\times 1600=100$$

Hence the frequency table is

Observed Frequency O	882	313	287	118
Expected Frequency E	900	300	300	100
$\frac{(O-E)^2}{E}$	0.36	0.563	0.563	3.24

Test statistic $\chi^2=\sum\frac{(O-E)^2}{E}=4.726$

Critical value of χ^2 at 5% level of significance for $\nu=(4-1)=3$ degrees of freedom is 7.815

Since calculated value of $\chi^2<\chi^2_{\alpha=0.05}$, accept the null hypothesis H_0.

5. Five coins are tossed 320 times. The number of heads observed is given below:

No. of Heads :	**0**	**1**	**2**	**3**	**4**	**5**
Frequency :	**15**	**45**	**85**	**95**	**60**	**20**

Examine whether the coin is unbiased. Use 5% L.O.S. **(QC 41313 MA 6452 MAY 18)**

The chance of getting a head in a single throw $p=\frac{1}{2}$. Hence $q=\frac{1}{2}$.

Given $n=5$ and $N=320$.

The expected frequencies $E(x)$ for x head(success) in N trials when n coins are tossed are given by $N\times nc_x p^x q^{n-x}$.

When 5 coins are tossed, the expected frequencies for 0, 1, 2, 3, 4, 5 heads in 320 trials are given by

$$E(0)=320\times 5c_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5=\frac{320}{32}=10 \qquad E(1)=320\times 5c_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4=\frac{1600}{32}=50$$

$$E(2)=320\times 5c_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3=\frac{3200}{32}=100 \qquad E(3)=320\times 5c_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{3200}{32}=100$$

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$E(4) = 320 \times 5c_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 = \frac{1600}{32} = 50$ $\quad$ $E(5) = 320 \times 5c_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = \frac{320}{32} = 10$

Observed Frequency – O	15	45	85	95	60	20
Expected Frequency – E	10	50	100	100	50	10
$\frac{(O-E)^2}{E}$	2.5	0.5	2.25	0.25	2	5

Null Hypothesis H_0: The coin is un biased. i.e. $p = \frac{1}{2}$

Test statistic $\psi^2 = \sum \frac{(O-E)^2}{E} = 12.5$

Critical value of χ^2 at 5% level of significance for $v = (6-1) = 5$ degrees of freedom is 11.07

Since calculated value of $\chi^2 > \chi^2_{\alpha=0.05}$, reject the null hypothesis H_0.
i.e. we can conclude that the coin is biased.

6. A survey of 320 families with 5 children each revealed the following information.

No. of Girls	**:**	**0**	**1**	**2**	**3**	**4**	**5**
No. of Boys	**:**	**5**	**4**	**3**	**2**	**1**	**0**
Frequency	**:**	**14**	**56**	**110**	**88**	**40**	**12**

Is this result consistent with the hypothesis that male and female births are equally probable?. Use 5% level of significance.

The probability for male birth $p = \frac{1}{2}$. Hence the probability for female birth $q = \frac{1}{2}$.
Given $n = 5$ and $N = 320$.

The probability for r male births in a family of 5 is $5c_r p^r q^{n-r}$.

In 320 families the number of r male births is $E(r) = 320 \times 5c_r p^r q^{n-r} = 320 \times 5c_r (0.5)^r (0.5)^{5-r}$

Therefore the expected frequencies of 0, 1, 2, 3, 4, 5 male births are given by

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$$E(0)=320\times 5c_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5=\frac{320}{32}=10 \qquad E(1)=320\times 5c_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4=\frac{1600}{32}=50$$

$$E(2)=320\times 5c_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3=\frac{3200}{32}=100 \qquad E(3)=320\times 5c_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{3200}{32}=100$$

$$E(4)=320\times 5c_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1=\frac{1600}{32}=50 \qquad E(5)=320\times 5c_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0=\frac{320}{32}=10$$

Observed Frequency O	14	56	110	88	40	12
Expected Frequency E	10	50	100	100	50	10
$\frac{(O-E)^2}{E}$	1.6	0.72	1	1.44	2	0.4

Null Hypothesis H_0: Male and female births are equally probable. i.e. $p=\frac{1}{2}$

Test statistic $\psi^2=\sum\frac{(O-E)^2}{E}=7.16$

Critical value of χ^2 at 5% level of significance for $\nu=(6-1)=5$ degrees of freedom is 11.07

Since calculated value of $\chi^2<\chi^2_{\alpha=0.05}$, accept the null hypothesis H_0.
i.e. the result consistent with the hypothesis that male and female births are equally probable

7. Fit a Poisson distribution for the following data and test the goodness of fit at 5% level of significance. **(QC 60045 MA 3251 APR 22)**

x:	**0**	**1**	**2**	**3**	**4**	**5**	**Total**
f:	**6**	**13**	**13**	**8**	**4**	**3**	**47**

Probability law for Poisson distribution is $P(x)=\frac{e^{-\lambda}\lambda^x}{x!}, x=0,1,2,.....$

x_i	f_i	$f_i x_i$
0	6	0
1	13	13
2	13	26

3	8	24
4	4	16
5	3	15
	N = 47	94

Mean of the distribution = $\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{94}{47} = 2$

To fit a Poisson distribution we require parameter $\lambda = \bar{x} = 2$.

By Poisson distribution the expected frequency of x successes is

$$E(x) = N \times P(x) = N \times \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,.....$$

$$E(0) = 47 \times \frac{e^{-2}(2)^0}{0!} = 6.36 \qquad E(1) = 47 \times \frac{e^{-2}(2)^1}{1!} = 12.7 \qquad E(2) = 47 \times \frac{e^{-2}(2)^2}{2!} = 12.7$$

$$E(3) = 47 \times \frac{e^{-2}(2)^3}{3!} = 8.48 \qquad E(4) = 47 \times \frac{e^{-2}(2)^4}{4!} = 4.24 \qquad E(5) = 47 \times \frac{e^{-2}(2)^5}{5!} = 1.69$$

Hence the frequency table is

X	0	1	2	3	4	5	Total
Observed Frequency O	6	13	13	8	4	3	47
Expected Frequency E	6.36	12.7	12.7	8.48	4.24	1.69	
$\frac{(O-E)^2}{E}$	0.02	0.007	0.007	0.027	0.013	1.01	1.084

H_0 : Poisson fit is a good fit for the data. H_1 : Poisson fit is a not good fit

Test statistic $\chi^2 = \sum \frac{(O-E)^2}{E} = 1.084$

Critical value of χ^2 at 5% level of significance for $\nu = 5-2 = 3$ degrees of freedom is 7.81

Since calculated value of $\chi^2 < \chi^2_{\alpha=0.05}$, accept the null hypothesis H_0.

Chi Square Test for Independence of Attributes

1. Give the formula for the χ^2 test of independence for $\begin{array}{|cc|} a & b \\ c & d \end{array}$. (QC 57506 MA 6452 MAY 16)

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

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Working Rule
Let A and B be two given attributes which are divided into r and s sub classes given in the form of contingency $r \times s$ table
• Set up null hypothesis H_0: The attributes A and B are independent. • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test. • Compute expected frequencies $E(a) = \frac{(row\ total \times column\ total)\ of\ 'a'}{grand\ total}$ • Compute test statistic $\chi^2 = \sum_{i=1}^{n} \frac{(O-E)^2}{E}$. • Choose appropriate level of significance and degrees of freedom $(r-1)(s-1)$ and find table value of χ^2_α (critical value) • Compare calculated value of $\lvert \chi^2 \rvert$ with the tabulated value. • Conclusion: If $\lvert \chi^2 \rvert < z_\alpha$ then accept H_0. If $\lvert \chi^2 \rvert > \chi^2_\alpha$ then reject H_0.

Solved Problems

1. **A total number of 3759 persons were interviewed in a public opinion survey on a political proposal. Of them, 1872 were men and the rest women. 2257 persons were favour of the proposal and 917 were opposed to it. 243 men were undecided and 442 women were opposed to the proposal. Justify or contradict the hypothesis that there is no association between sex of persons and their attitude at 5% level of significance. (QC 60045 MA 3251 APR 22)**

From the given data, we have the following observed frequency table:

	Favoured	Opposed	Undecided	Total
Men	1154	475	243	1872
Women	1103	442	342	1887
Total	2257	917	585	3759

Expected frequencies can be evaluated as follows:

	Favoured	Opposed	Undecided
Men	$\frac{2257 \times 1872}{3759} = 1124$	$\frac{917 \times 1872}{3759} = 457$	$\frac{585 \times 1872}{3759} = 291$

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Women	$\frac{2257\times1887}{3759}=1133$	$\frac{917\times1887}{3759}=460$	$\frac{585\times1887}{3759}=294$

Null hypothesis H_0: Sex and attitude are independent

Alternative Hypothesis H_1: Sex and attitude are not independent

Observed Frequency O	1154	475	243	1103	442	342
Expected Frequency E	1124	457	291	1133	460	294
$\frac{(O-E)^2}{E}$	0.8	0.71	7.92	0.79	0.7	7.84

Test statistic $\chi^2 = \sum \frac{(O-E)^2}{E} = 18.76$

Critical value of χ^2 at 5% level of significance for $\nu=(3-1)(2-1)=2$ degrees of freedom is 5.99

Since calculated value of $\chi^2 > \chi^2_{\alpha=0.05}$, reject the null hypothesis H_0.

2. Mechanical engineers testing a new arc welding technique, classified welds both with respect to appearance and an X-ray inspection.

		Appearance		
		Bad	**Normal**	**Good**
X-Ray	**Bad**	**20**	**7**	**3**
	Normal	**13**	**51**	**16**
	Good	**7**	**12**	**21**

Test for independence using 0.05 level of Significance.

(QC 41313 MA 6452 MAY 18)

From the given data, we have the following observed frequency table:

Appearance→ X-Ray↓	Bad	Normal	Good	Total
Bad	20	7	3	30
Normal	13	51	16	80
Good	7	12	21	40
Total	40	70	40	150

Expected frequencies can be evaluated as follows:

Appearance→ X-Ray↓	Bad	Normal	Good
Bad	$\frac{40\times 30}{150}=8$	$\frac{70\times 30}{150}=14$	$\frac{40\times 30}{150}=8$
Normal	$\frac{40\times 80}{150}=21.33$	$\frac{70\times 80}{150}=37.33$	$\frac{40\times 80}{150}=21.33$
Good	$\frac{40\times 40}{150}=10.66$	$\frac{70\times 40}{150}=18.66$	$\frac{40\times 40}{150}=10.66$

Null hypothesis H_0: X-ray inspection and appearance test are independent

Alternative Hypothesis H_1: X-ray inspection and appearance test not independent

Observed Frequency O	20	7	3	13	51	16	7	12	21
Expected Frequency E	8	14	8	21.33	37.33	21.33	10.66	18.66	10.66
$\frac{(O-E)^2}{E}$	18	3.5	3.125	3.25	5	1.33	1.256	2.377	10.02

Test statistic $\psi^2=\sum\frac{(O-E)^2}{E}=47.858$

Critical value of χ^2 at 5% level of significance for $\nu=(3-1)(3-1)=4$ degrees of freedom is 9.488

Since calculated value of $\chi^2 > {\chi^2}_{\alpha=0.05}$, reject the null hypothesis H_0.

3. **Using the data given in the following table to test at the 0.01 level of significance whether a person's ability in Mathematics is independent of his/her interest in Statistics.**

		Ability in Mathematics		
		Low	**Average**	**High**
	Low	**63**	**42**	**15**
Interest in Statistics	**Average**	**58**	**61**	**31**
	High	**14**	**47**	**29**

(QC 20753 MA 6452 NOV 18)

From the given data, we have the following observed frequency table:

Ability→ Interest↓	Low	Average	High	Total
Low	63	42	15	120
Average	58	61	31	150
High	14	47	29	90
Total	135	150	75	360

Expected frequencies can be evaluated as follows:

Ability→ Interest↓	Low	Average	High
Low	$\frac{135\times120}{360}=45$	$\frac{150\times120}{360}=50$	$\frac{75\times120}{360}=25$
Average	$\frac{135\times150}{360}=56.25$	$\frac{150\times150}{360}=62.5$	$\frac{75\times150}{360}=31.25$
High	$\frac{135\times90}{360}=33.75$	$\frac{150\times90}{360}=37.5$	$\frac{75\times90}{360}=18.75$

Null hypothesis H_0: Ability in Mathematics and interest in Statistics are independent

Alternative Hypothesis H_1: Ability in Mathematics and interest in Statistics are not independent

Observed Frequency O	63	42	15	58	61	31	14	47	29

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Expected Frequency E	45	50	25	56.25	62.5	31.25	33.75	37.5	18.75
$\frac{(O-E)^2}{E}$	7.2	1.28	0.25	0.054	0.036	0.002	11.55	2.41	5.6

Test statistic $\chi^2 = \sum \frac{(O-E)^2}{E} = 28.38$

Critical value of χ^2 at 1% level of significance for $\nu = (3-1)(3-1) = 4$ degrees of freedom is 13.27

Since calculated value of $\chi^2 > \chi^2_{\alpha=0.01}$, reject the null hypothesis H_0.

4. **Two groups of 100 people each were taken for testing the use of a vaccine, 15% contracted the disease out of inoculated persons, while 25 contracted the disease in the other group. Test the efficacy of the vaccine using χ^2 test.**

	Affected	**Not Affected**	**Total**
Inoculated	**15**	**85**	**100**
Not Inoculated	**25**	**75**	**100**
Total	**40**	**160**	**200**

Null hypothesis H_0: The vaccine is not effective

Alternative Hypothesis H_1: The vaccine is effective

Test statistic $\chi^2 = \frac{N(ad-bc)^2}{(a+b)(c+d)(a+c)(b+d)} = \frac{200(15\times 75 - 25\times 85)}{(15+85)(25+75)(15+25)(85+75)} = 3.125$

Critical value of χ^2 at 5% level of significance for $\nu = (2-1)(2-1) = 1$ degrees of freedom is 3.841

Since calculated value of $\chi^2 < \chi^2_{\alpha=0.05}$, accept the null hypothesis H_0 and conclude that the vaccine is not effective.

F-Test : Test of Significance for Equality of Population Variance (Small Sample)

This is used to test the significance of sample estimates of population variance. Under	Write about $F-$ test. MA 6452 NOV 15

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the null hypothesis that the population variances are equal, the test statistic is given by $F = \frac{S_1^2}{S_2^2}$, assuming $S_1^2 > S_2^2$ where $S_1^2 = \frac{1}{n_1 - 1}\sum(x - \bar{x})^2$, $S_2^2 = \frac{1}{n_2 - 1}\sum(y - \bar{y})^2$ are unbiased estimates of the common population variance σ^2 obtained from two independent samples. The test statistic follows F-distribution with degrees freedom $(n_1 - 1, n_2 - 1)$. By comparing the calculated value, with the tabulated value for the above degrees of freedom at specific level of significance, the null hypothesis is either accepted or rejected.	

2. **Write any two important uses of normal curve.** **(QC 53250 MA 6452 MAY 19)**

Many of the distributions of sample statistic tend to normality for large samples and as such they can best be studied with the help of the normal curves.

Theory of normal curves can be applied to the graduation of the curves which are not normal

Test of significance for equality of population variances **Working Rule**
Let A and B be two samples with sizes n_1 and n_2 and SD s_1 and s_2
• Set up null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ The Population variances are same. • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test. • Compute estimated population variance $S_1^2 = \frac{1}{n_1 - 1}\sum(x - \bar{x})^2$ and $S_2^2 = \frac{1}{n_2 - 1}\sum(y - \bar{y})^2$ • Compute test statistic $F = \frac{S_1^2}{S_2^2}$, assuming $S_1^2 > S_2^2$. • Choose appropriate level of significance and degrees of freedom $(n_1 - 1, n_2 - 1)$ and find table value of F_α (critical value) • Compare calculated value of $\lvert F \rvert$ with the tabulated value. • Conclusion: If $\lvert F \rvert < F_\alpha$ then accept H_0. If $\lvert F \rvert > F_\alpha$ then reject H_0.

Solved Problems

1. Test if the variances are significantly different for: **(QC 27331 MA 6452 NOV 15)**

x_1	:	**24**	**27**	**26**	**21**	**25**	
x_2	:	**27**	**30**	**32**	**36**	**28**	**23**

Here we have to apply F test. Given $n_1 = 5$ *and* $n_2 = 6$

x_1	24	27	26	21	25		$\sum x_1 = 123$
x_1^2	576	729	676	441	625		$\sum x_1^2 = 3047$
x_2	27	30	32	36	28	23	$\sum x_2 = 176$
x_2^2	729	900	1024	1296	784	529	$\sum x_2^2 = 5262$

Mean of sample 1 : $\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{123}{5} = 24.6$

Variance of sample 1 : $s_1^2 = \frac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \frac{3047}{5} - 24.6^2 = 4.24$

Mean of sample 2 : $\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{176}{6} = 29.3$

Variance of sample 2 : $s_2^2 = \frac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \frac{5262}{6} - 29.3^2 = 18.51$

To test the variance

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample 1 and 2)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Estimated Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{5 \times 4.24}{4} = 5.3$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{6 \times 18.51}{5} = 22.21$

Test statistic $F = \frac{S_2^2}{S_1^2} = \frac{22.21}{5.3} = 4.19$

Degrees of freedom $(n_2 - 1, n_1 - 1) = (5, 4)$

Table value of F for degrees $(5, 4)$ of freedom at 5% level of significance is 6.26

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. two sample variances do not differ significantly.

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2 Pumpkins were grown under two experimental conditions. Two random samples of 11 and 9 pumpkins show the sample standard deviations of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, test the hypothesis that the true variances are equal, against the alternative hypothesis that they are not at the 10% level of significance. **(QC 20753 MA 6452 NOV 18)**

Given $n_1 = 11, \; n_2 = 9, \quad s_1 = 0.8, \quad s_2 = 0.5$

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (Populations variances are equal)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{11 \times 0.8^2}{10} = 0.704$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{9 \times 0.5^2}{8} = 0.28$

Test statistic $F = \frac{S_1^2}{S_2^2} = \frac{0.704}{0.28} = 2.5$

Degrees of freedom $(n_1 - 1, n_2 - 1) = (10, 8)$

Table value of F for degrees $(10, 8)$ of freedom at 10% level of significance is 5.81

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. difference between population variances are not significant.

3. A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded the following increase in weight.

Diet A :	**5**	**6**	**8**	**1**	**12**	**4**	**3**	**9**	**6**	**10**
Diet B :	**2**	**3**	**6**	**8**	**10**	**1**	**2**	**8**		

Find the variances are significantly different. **(QC 80610 MA 6452 NOV 16)**

Here we have to apply F test. Given $n_1 = 10$ *and* $n_2 = 8$

x_1	6	6	8	1	12	4	3	9	6	10	$\sum x_1 = 65$
x_1^2	36	36	64	1	144	16	9	81	36	100	$\sum x_1^2 = 523$
x_2	2	3	6	8	10	1	2	8			$\sum x_2 = 40$
x_2^2	4	9	36	64	100	1	4	64			$\sum x_2^2 = 282$

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Mean of Diet A : $\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{65}{10} = 6.5$

Variance of Diet A : $s_1^2 = \frac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \frac{523}{10} - 6.5^2 = 10.05$

Mean of Diet B : $\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{40}{8} = 5$

Variance of Diet B : $s_2^2 = \frac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \frac{282}{8} - 5^2 = 10.25$

To test the variance

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample 1 and 2)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10 \times 10.05}{9} = 11.16$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{8 \times 10.25}{7} = 11.71$

Test statistic $F = \frac{s_2^2}{s_1^2} = \frac{11.71}{11.16} = 1.04$

Degrees of freedom $(n_2 - 1, n_1 - 1) = (7, 9)$

Table value of F for degrees $(7, 9)$ of freedom at 5% level of significance is 3.29

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. two population variance do not differ significantly.

4. **Time taken by workers in performing a job is given below:**

Method 1	**20**	**16**	**26**	**27**	**23**	**22**
Method 2	**27**	**33**	**42**	**35**	**34**	**38**

Test whether there is any significant difference between the variances of the time distribution at 5% level of significance. **(QC 50782 MA 6452 NOV 17)**

Here we have to apply F test. Given $n_1 = 6$ and $n_2 = 6$

x_1	20	16	26	27	23	22	$\sum x_1 = 134$

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x_1^2	400	256	676	729	529	484	$\sum x_1^2 = 3074$
x_2	27	33	42	35	34	38	$\sum x_2 = 209$
x_2^2	729	1089	1764	1225	1156	1444	$\sum x_2^2 = 7407$

Mean of method 1 : $\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{134}{6} = 22.3$

Variance of method 1 : $s_1^2 = \frac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \frac{3074}{6} - 22.3^2 = 15.04$

Mean of method 2 : $\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{209}{6} = 34.8$

Variance of method 2 : $s_2^2 = \frac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \frac{7407}{6} - 34.8^2 = 23.46$

To test the variance

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample 1 and 2)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{6 \times 15.04}{5} = 18.04$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{6 \times 23.46}{5} = 28.15$

Test statistic $F = \frac{S_2^2}{S_1^2} = \frac{28.15}{18.04} = 1.56$

Degrees of freedom $(n_2 - 1, n_1 - 1) = (5, 5)$

Table value of F for degrees $(5, 5)$ of freedom at 5% level of significance is 10.97

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. two sample variances do not differ significantly.

5. **Two independent samples of sizes 9 and 7 from a normal population had the following values of the variables.**

Sample 1	**18**	**13**	**12**	**15**	**12**	**14**	**16**	**14**	**15**
Sample 2	**16**	**19**	**13**	**16**	**18**	**13**	**15**		

Do the estimates of the population variance differ significantly at 5% level of significance?
(QC 72071 MA 6452 MAY 17)

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Here we have to apply F test. Given $n_1 = 9 \ and \ n_2 = 7$

x_1	18	13	12	15	12	14	16	14	15	$\sum x_1 = 129$
x_1^2	324	169	144	225	144	196	256	196	225	$\sum x_1^2 = 1879$
x_2	16	19	13	16	18	13	15			$\sum x_2 = 110$
x_2^2	256	361	169	256	324	169	225			$\sum x_2^2 = 1760$

Mean of sample 1 : $\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{129}{9} = 14.3$

Variance of sample 1 : $s_1^2 = \frac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \frac{1879}{9} - 14.3^2 = 4.28$

Mean of sample 2 : $\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{110}{7} = 15.7$

Variance of sample 2 : $s_2^2 = \frac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \frac{1760}{7} - 15.7^2 = 4.94$

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample 1 and 2)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9 \times 4.28}{8} = 4.815$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{7 \times 4.94}{6} = 5.763$

Test statistic $F = \frac{S_2^2}{S_1^2} = \frac{5.763}{4.815} = 1.19$

Degrees of freedom $(n_2 - 1, n_1 - 1) = (6, 8)$

Table value of F for degrees $(6, 8)$ of freedom at 5% level of significance is 3.58

Since, calculated value of $|F| <$ the tabulated value, we accept H_0.

i.e. two population variance do not differ significantly.

Test of Significance of the Mean (t-test) – Small Sample

This $t-$distribution is used when sample size is ≤ 30 and the population SD is unknown.

To Test whether sample mean differs from the hypothetical population mean **Working Rule**
Sample size n, mean $\bar{x}$ **, SD** s **and population mean** μ **is given**
• Set up null hypothesis $H_0: \mu =$ given value. • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test. • Compute $t = \dfrac{\bar{x}-\mu}{\left(\dfrac{s}{\sqrt{n-1}}\right)}$. • If set of sample values are given find $\bar{x} = \dfrac{\sum x}{n}$ and $s = \sqrt{\dfrac{\sum (x-\bar{x})^2}{n-1}}$, then $t = \dfrac{\bar{x}-\mu}{\left(\dfrac{s}{\sqrt{n}}\right)}$ • Choose appropriate level of significance and degrees of freedom $(n-1)$ and find table value (critical value) of t_α. • Compare calculated value of $\lvert t \rvert$ with the tabulated value. • Conclusion: If $\lvert t \rvert < t_\alpha$ then accept H_0. If $\lvert t \rvert > t_\alpha$ then reject H_0.

Solved Problems

1 What are the applications of t-distributions? **(QC 11395 MA 2266 MAY 11)**

To test if the sample mean differs significantly from the population mean
To test the significance between two sample means.

2. For the following case, specify which probability distribution to use in a hypothesis test.

$(a).\ H_0: \mu = 27,\ H_1: \mu \neq 27,\ \bar{x} = 20.1,\ \sigma = 5,\ n = 12$ **(QC 41313 MA 6452 MAY 18)**

(a) Test of significance for single mean, small sample, two tailed test.

3. A company claims that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does

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this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.
(QC 20817 MA 8452 APR 22)

Given $n = 12$, $\mu = 46$ KW , $s = 11.9$ and $\bar{x} = 42$

1. $H_0 : \mu = 46$
2. $H_1 : \mu < 46$
3. $\alpha = 5\%$, $d.f = n - 1 = 12 - 1 = 11$
4. The test statistic $t = \dfrac{\bar{x} - \mu}{\left(\dfrac{s}{\sqrt{n-1}}\right)} = \dfrac{42 - 46}{\left(\dfrac{11.9}{\sqrt{12-1}}\right)} = -1.1$
5. For 5% level of significance, the tabulated value at 11 degrees freedom is $t = 2.2$
6. Conclusion : Since, calculated value of $|t| = 1.1 <$ the tabulated value, we accept H_0.

4. Machinist is making engine parts with axle diameters of 0.7 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.04 inch. Compute the statistic to test the work is meeting the specification. **(QC 53250 MA 6452 MAY 19)**

Given $\bar{x} = 0.742,\ n = 10$ and $s = 0.4$ and population mean $\mu = 0.7$

Null Hypothesis $H_0 : \mu = 0.7$ (the product is confirming the specification)

Alternative Hypothesis $H_1 : \mu \neq 0.7$ (two tailed test)

Test statistic $t = \dfrac{\bar{x} - \mu}{\dfrac{s}{\sqrt{n-1}}} = \dfrac{0.742 - 0.7}{\dfrac{0.4}{\sqrt{9}}} = \dfrac{0.042}{0.13} = 0.315$

Table value of t for $n - 1 = 9$ degrees freedom at 5% level of significance is 2.26

Since calculated value of $|t| <$ the tabulated value, we accept the null hypothesis.

5. Given a sample mean of 83, a sample standard deviation of 12.5 and a sample size of 22, test the hypothesis that the value of the population mean is 70 against the alternative that is more than 70. Use the 0.025 significance level. **(QC 41313 MA 6452 MAY 18)**

Given $\bar{x} = 83,\ n = 22$ and $s = 12.5$ and population mean $\mu = 70$

Null Hypothesis $H_0 : \mu = 70$ (sample mean is not different from the population mean)

Alternative Hypothesis $H_1 : \mu > 70$ (one tailed test)

Test statistic $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n-1}}} = \frac{83-70}{\frac{12.5}{\sqrt{21}}} = \frac{13}{0.841} = 2.727$

Table value of t for $n-1=21$ degrees freedom at 0.025 level of significance is 35.47

Since calculated value of $|t| <$ the tabulated value, we accept the null hypothesis.

6. A certain pesticide is packed into bags by a machine. A random sample of 10 bags is chosen and the contents of the bags is found to have the following weights (in kgs) 50, 49, 52, 44, 45, 48, 46, 45, 49 and 45. Test if the average quantity packed be taken as 50 kg.

(QC 27331 MA 6452 NOV 15)

Let us tabulate the values as follows:

x	50	49	52	44	45	48	46	45	49	45	**T: 473**
$(x-\bar{x})^2$	7.29	2.89	22.09	10.89	5.29	0.49	1.69	5.29	2.89	5.29	**T: 64.1**

$\bar{x} = \frac{\sum x}{n} = \frac{473}{10} = 47.3$ and $s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{64.1}{9}} = 2.66$

Null Hypothesis $H_0 : \mu = 50$ (sample mean weight is not different from the expected weight)

Alternative Hypothesis $H_1 : \mu \neq 50$ (two tailed test)

Test statistic $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} = \frac{47.3-50}{\frac{2.66}{\sqrt{10}}} = \frac{-2.7}{0.841} = -3.2$

Table value of t for $n-1=9$ degrees freedom at 5% level of significance is 2.26

Since calculated value of $|t| >$ the tabulated value, we reject the null hypothesis.

7. A random sample of 10 boys has the following IQ's 70, 83, 88, 95, 98, 100, 101, 107, 110 and 120. Do these data support the assumption of a population mean IQ of 100 at 5% level of significance? (QC 50782 MA 6452 NOV 17)

Let us tabulate the values as follows:

x	70	83	88	95	98	100	101	107	110	120	**T: 972**
$(x-\bar{x})^2$	739.8	201.6	84.6	4.84	0.64	7.84	14.4	96.1	163.8	519.8	**T: 1833.42**

$$\bar{x} = \frac{\sum x}{n} = \frac{972}{10} = 97.2 \quad \text{and} \quad s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{1833.42}{9}} = 14.27$$

Null Hypothesis $H_0 : \mu = 100$ (no difference between sample and population mean)

Alternative Hypothesis $H_1 : \mu \neq 100$ (two tailed test)

Test statistic $t = \dfrac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} = \dfrac{97.2-100}{\frac{14.27}{\sqrt{10}}} = \dfrac{-2.8}{4.51} = -0.62$

Table value of t for $n-1=9$ degrees freedom at 5% level of significance is 2.26

Since calculated value of $|t| <$ the tabulated value, we accept the null hypothesis.

Testing of Significance for Single Mean – Large Sample

To Test whether sample mean differs from the hypothetical population mean **Working Rule**
Sample size n, mean $\bar{x}$ **, SD** S **and population mean** μ, SD σ **is given**
• Set up null hypothesis $H_0 : \mu = \bar{x}$ (the sample is drawn from the given population) • Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test.

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- Compute $z = \frac{\bar{x}-\mu}{\left(\frac{S}{\sqrt{n}}\right)}$ (or) $z = \frac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$.
- Choose appropriate level of significance and find table value of z_α (critical value)
- Compare calculated value of $|z|$ with the tabulated value.
- Conclusion: If $|z| < z_\alpha$ then accept H_0. If $|z| > z_\alpha$ then reject H_0.

Solved Problems

1. For the following case, specify which probability distribution to use in a hypothesis test.

(a). $H_0 : \mu = 98,\ H_1 : \mu > 98,\ \bar{x} = 65,\ s = 12,\ n = 42$ **(QC 41313 MA 6452 MAY 18)**

(a) Test of significance for single mean, large sample, one tailed test

2. A standard sample of 200 tins of coconut oil gave an average weight of 4.95 kgs with a standard deviation of 0.21 kg. Do we accept that the net weight is 5kgs per tin at 5% level of significance? **(QC 72071 MA 6452 MAY 17)**

Given $n = 200,\quad \bar{x} = 4.95,\quad s = 0.21,\quad \mu = 5$

Null Hypothesis $H_0 : \mu = 5$ (no significant difference between sample mean and population mean)

Alternative Hypothesis $H_1 : \mu \neq 5$ (two tailed test)

Test statistic $z = \frac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}} = \frac{4.95-5}{\frac{0.21}{\sqrt{200}}} = -3.36$

Table value of z for 5% level of significance is 1.96

Since calculated value of $|z| >$ the tabulated value, we reject the null hypothesis.

we can't accept that the net weight is 5kgs per tin

3. A random sample of 100 recorded deaths in India during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance. **(QC 20817 MA 8452 APR 22)**

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Given $n = 100$, $\bar{x} = 71.8$, $\mu = 70$, $\sigma = 8.9$

1. $H_0 : \mu = 70$

2. $H_1 : \mu > 70$ [use *right* tailed test]

3. $\alpha = 5\%$

4. The test statistic $Z = \dfrac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{71.8-70}{\left(\frac{8.9}{\sqrt{100}}\right)} = 2.02$

5. Tabulated value of z at 5% level is $|z_\alpha| = 1.64$

6. Conclusion: Since calculated value of $|Z| = 2.02$ > the tabulated value, we reject the null hypothesis H_0. i.e. there is significant difference between sample mean and population mean.

4. A sample of 900 members has a mean 3.4 and standard deviation 2.61 cms. Is the sample from a large population of mean 3.25 cms and standard deviation 2.61 cms.

(QC 80610 MA 6452 NOV 16)

Given $n = 900$, $\bar{x} = 3.4$, $s = 2.61$, $\mu = 3.25$, $\sigma = 2.61$

Null Hypothesis $H_0 : \mu = 3.25$ (no significant difference between sample mean and population mean)

Alternative Hypothesis $H_1 : \mu \neq 3.25$ (two tailed test)

Test statistic $z = \dfrac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{3.4-3.25}{\frac{2.61}{\sqrt{900}}} = 1.724$

Table value of z for 5% level of significance is 1.96

Since calculated value of $|z|$ < the tabulated value, we accept the null hypothesis.

Test for Difference of means of two samples – t test – Small Samples

Test of significance of the difference between two sample means **Working Rule**
When $n_1, \bar{x}, s_1$ be the sample size, mean and SD of first sample and $n_2, \bar{y}, s_2$ be the sample size, mean and SD of second sample is given

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- Set up null hypothesis $H_0 = \mu_1 = \mu_2$ (samples are drawn from the populations with same mean)
- Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test.
- Compute test statistic $t = \frac{\bar{x} - \bar{y}}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where estimated population variance $S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.
- Choose appropriate level of significance, degrees of freedom $n_1 + n_2 - 2$ and table value of t_α (critical value)
- Compare calculated value of $|t|$ with the tabulated value.
- Conclusion: If $|t| < t_\alpha$ then accept H_0. If $|t| > t_\alpha$ then reject H_0.

Note 1: If we were asked to test whether both the samples come from same normal population, we have to apply both t and F tests.

Note 2: Instead of sample values x_i , y_i sometimes, the difference between them, say, $X = x_i - y_i$ will be given. In that case the test statistic is $t = \frac{\bar{X}}{\frac{S}{\sqrt{n}}}$ and proceed like test of hypothesis of single mean – Small sample problem.

Note 3: Instead of two different samples, pairs of values which are correlated will be given. Then the test statistic is $t = \frac{\bar{d}}{\frac{S}{\sqrt{n}}}$ where $d = x_i - y_i$

Solved Problems

1. Two independent samples of sizes 8 and 7 contained the following values:

Sample I :	**19**	**17**	**15**	**21**	**16**	**18**	**16**	**14**
Sample II:	**15**	**14**	**15**	**19**	**15**	**18**	**16**	

(QC 60045 MA 3251 APR 22)

Is the difference between the sample means significance? Use 5% level of significance.

Given $n_1 = 8$ and $n_2 = 7$

x_1	19	17	15	21	16	18	16	14	$\sum x_1 = 136$
x_1^2	361	289	225	441	256	324	256	196	$\sum x_1^2 = 2348$
x_2	15	14	15	19	15	18	16		$\sum x_2 = 112$
x_2^2	225	196	225	361	225	324	256		$\sum x_2^2 = 1812$

Mean of first sample $\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{136}{8} = 17$

Variance of I sample $s_1^2 = \frac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \frac{2348}{8} - 17^2 = 4.5$

Mean of second sample $\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{112}{7} = 16$

Variance of II sample $s_2^2 = \frac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \frac{1812}{7} - 16^2 = 2.85$

Population variance $S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{8 \times 4.5 + 7 \times 2.85}{8 + 7 - 2} = 4.303$ and hence $S = 2.07$

Null hypothesis $H_0 = \mu_1 = \mu_2$ (No significant difference between means of sample I and II)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{17 - 16}{2.07\sqrt{\left(\frac{1}{8} + \frac{1}{7}\right)}} = \frac{1}{1.075} = 0.93$

Table value of t at 5% level of significance for $v = n_1 + n_2 - 2 = 13$ degrees freedom is $t_{0.05} = 2.16$

Since, calculated value of $|t| <$ the tabulated value, we accept H_0

i.e. two sample means do not differ significantly.

2. **The nicotine content in milligram of 2 samples of tobacco were found to be as follows:**

Sample A	**:**	**24**	**27**	**26**	**21**	**25**	
Sample B	**:**	**27**	**30**	**28**	**31**	**22**	**36**

Can it be said that these samples were from normal population with the same mean?

Here we have to apply both t and F test.

To test the mean

Given $n_1 = 5 \ \ and \ \ n_2 = 6$

x_1	24	27	27	21	25		$\sum x_1 = 124$
x_1^2	576	729	729	441	625		$\sum x_1^2 = 3100$
x_2	27	30	28	31	22	36	$\sum x_2 = 174$
x_2^2	729	900	784	961	484	1296	$\sum x_2^2 = 5154$

Mean of sample A : $\bar{x}_1 = \dfrac{\sum x_1}{n_1} = \dfrac{124}{5} = 24.8$

Variance of sample A : $s_1^2 = \dfrac{\sum x_1^2}{n_1} - (\bar{x}_1)^2 = \dfrac{3100}{5} - 24.8^2 = 4.96$

Mean of sample B : $\bar{x}_2 = \dfrac{\sum x_2}{n_2} = \dfrac{174}{6} = 29$

Variance of sample B : $s_2^2 = \dfrac{\sum x_2^2}{n_2} - (\bar{x}_2)^2 = \dfrac{5154}{6} - 29^2 = 18$

Population variance $S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \dfrac{5 \times 4.96 + 6 \times 18}{5 + 6 - 2} = 14.75$ and hence $S = 3.84$

Null hypothesis $H_0 = \mu_1 = \mu_2$ (No significant difference between means of sample A and B)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $t = \dfrac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \dfrac{24.8 - 29}{3.84\sqrt{\left(\dfrac{1}{5} + \dfrac{1}{6}\right)}} = \dfrac{-4.2}{2.326} = -1.8$

Table value of t at 5% level of significance for $v = n_1 + n_2 - 2 = 9$ degrees freedom is $t_{0.05} = 2.262$

Since, calculated value of $|t| <$ the tabulated value, we accept H_0

i.e. two sample means do not differ significantly and both the samples come from same population.

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To test the variance

Null hypothesis $H_0: \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample A and B)

Alternative Hypothesis: $H_1: \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{5 \times 4.96}{4} = 6.2$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{6 \times 18}{5} = 21.6$

Test statistic $F = \frac{S_2^2}{S_1^2} = \frac{21.6}{6.2} = 3.48$

Degrees of freedom $(n_2 - 1, n_1 - 1) = (5, 4)$

Table value of F for degrees $(5, 4)$ of freedom at 5% level of significance is 6.26

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. two sample variances do not differ significantly.

Therefore we conclude that both the samples were from same normal population with same mean.

3. **Two random samples gave the following results:** **(QC 80610 MA 6452 NOV 16)**

Sample	Size	Sample Mean	Sum of squares of deviation from the mean
1	**10**	**15**	**90**
2	**12**	**14**	**108**

Test whether the samples come from the same normal population at 5% level of significance (given $F_{0.05}(9,11) = 2.9,\ F_{0.05}(11,9) = 3.1,\ t_{0.05}(20) = 2.086,\ t_{0.05}(22) = 2.07$ approximately)

Here we have to apply both t and F test.

To test the mean

Given sample sizes $n_1 = 10$ *and* $n_2 = 12$

Sample mean $\bar{x}_1 = 15$ and $\bar{x}_2 = 14$

Also given that $\sum (x_1 - \bar{x}_1)^2 = 90$ and $\sum (x_2 - \bar{x}_2)^2 = 108$

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$\therefore$ sample variances are $s_1^2 = \frac{1}{n_1 - 1}\sum\left(x_1 - \overline{x}_1\right)^2 = \frac{1}{9}(90) = 10$ and

$$s_2^2 = \frac{1}{n_2 - 1}\sum\left(x_2 - \overline{x}_2\right)^2 = \frac{1}{11}(108) = 9.8$$

Population variance $S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10\times 10 + 12\times 9.8}{10 + 12 - 2} = 10.88$ and hence $S = 3.298$

Null hypothesis $H_0 = \mu_1 = \mu_2$ (No significant difference between means of sample 1 and 2)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $t = \frac{\overline{x}_1 - \overline{x}_2}{S\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 14}{3.298\sqrt{\left(\frac{1}{10} + \frac{1}{12}\right)}} = \frac{1}{1.411} = 0.708$

Table value of t at 5% level of significance for $\nu = n_1 + n_2 - 2 = 20$ degrees freedom is $t_{0.05} = 2.086$

Since, calculated value of $|t| <$ the tabulated value, we accept H_0

i.e. two sample means do not differ significantly. and both the samples come from same population.

To test the variance

Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ (No significant difference between variances of sample 1 and 2)

Alternative Hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

Population variances are $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10\times 10}{9} = 11.11$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{12\times 9.8}{11} = 10.69$

Test statistic $F = \frac{S_1^2}{S_2^2} = \frac{11.11}{10.69} = 1.03$

Degrees of freedom $\left(n_1 - 1, n_2 - 1\right) = \left(9, 11\right)$

Table value of F for degrees $\left(9, 11\right)$ of freedom at 5% level of significance is 2.9

Since, calculated value of $|F| <$ the tabulated value, we accept H_0

i.e. two sample variances do not differ significantly.

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Therefore we conclude that both the samples were from same normal population with same mean.

4. **A certain medicine administered to each of 10 patients resulted in the following increases in the B.P. 8, 8, 7, 5, 4, 1, 0, 0, –1, –1. Can it be concluded that the medicine was responsible for the increase in B.P. 5% level of significance.** **(QC 72071 MA 6452 MAY 17)**

We are given the increments in blood pressure i.e $X = x_i - y_i$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (no significant difference in the BP before and after the medicine)

Alternative Hypothesis $H_1 : \mu_1 < \mu_2$ (one tailed test)

Let us tabulate the values as follows:

X	8	8	7	5	4	1	0	0	–1	–1	**T: 31**
$(X - \bar{X})^2$	24.1	24.1	15.21	3.61	0.81	4.41	9.61	9.61	16.81	16.81	**T: 125.08**

$$\bar{X} = \frac{\sum X}{n} = \frac{31}{10} = 3.1 \quad \text{and} \quad S = \sqrt{\frac{\sum (X - \bar{X})^2}{n-1}} = \sqrt{\frac{125.08}{9}} = 3.727$$

Test statistic $t = \dfrac{\bar{X}}{\frac{S}{\sqrt{n}}} = \dfrac{3.1}{\frac{3.727}{\sqrt{10}}} = \dfrac{3.1}{1.178} = 2.63$

Table value of t for $n-1=9$ degrees freedom at 5% level of significance is 1.83

Since calculated value of $|t| >$ the tabulated value, we reject the null hypothesis.
(i.e. the medicine was responsible for the increase in B.P.)

5. **Memory capacity of 9 students was tested before and after a meditation treatment for a month. State whether the treatment was effective or not from the following data:**

Before treatment	**:**	**10**	**15**	**9**	**3**	**7**	**12**	**16**	**17**	**4**
After treatment	**:**	**12**	**17**	**8**	**5**	**6**	**11**	**18**	**20**	**3**

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We are given the paired values i.e. same set of students and the data are concerned.

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (training was not effective)

Alternative Hypothesis $H_1 = \mu_1 \neq \mu_2$

Let us tabulate the values as follows:

Before Training	10	15	9	3	7	12	16	17	4	
After Training	12	17	8	5	6	11	18	20	3	
Difference d	2	2	–1	2	–1	–1	2	3	–1	$\sum d = 7$
d^2	4	4	1	4	1	1	4	9	1	$\sum d^2 = 29$

$\bar{d} = \dfrac{\sum d}{n} = \dfrac{7}{9} = 0.7778$ and $S = \sqrt{\dfrac{\sum d^2}{n-1}} = \sqrt{\dfrac{29}{8}} = 1.9$

Test statistic $t = \dfrac{\bar{d}}{\frac{S}{\sqrt{n}}} = \dfrac{0.778}{\frac{1.9}{\sqrt{9}}} = 1.23$

Table value of t for $n-1=8$ degrees freedom at 5% level of significance is 2.31

Since calculated value of $|t| <$ the tabulated value, we accept the null hypothesis. i.e. training was not improving the memory capacity

Test of significance for difference of means – Large Sample

Working Rule
When $n_1, \bar{x}_1, s_1$ be the sample size, mean and SD of first sample and $n_2, \bar{x}_2, s_2$ be the sample size, mean and SD of second sample and population SD σ is given

- Set up null hypothesis $H_0 = \mu_1 = \mu_2$ (samples are drawn from the populations with same mean)
- Set up alternative hypothesis H_1. This will determine whether we have to use right tailed or left tailed or two tailed test.
- Compute test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$ (if population SD not known and distinct) or

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$ (if samples are drawn from same population) (or)

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}}$$ (if samples are drawn from two population with same SD)

- Choose appropriate level of significance and find table value of z_α (critical value)
- Compare calculated value of $|z|$ with the tabulated value.
- Conclusion: If $|z| < z_\alpha$ then accept H_0. If $|z| > z_\alpha$ then reject H_0.

Solved Problems

1 **Write down the formula of test statistic t to the significance of difference between the mean(large samples)** **(QC 80610 MA 6452 NOV 16)**

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$ or $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$

2. **The sales manager of a large company conducted a sample survey in states A and B taking 400 samples in each case. The results were**

	State A	**State B**
Average Sales	**Rs.2,500**	**Rs.2,200**
S.D.	**Rs.400**	**Rs.550**

Test whether the average sales is the same in the 2 states at 1% level of significance.
(QC 72071 MA 6452 MAY 17)

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Here $n_1 = 400,\ \bar{x}_1 = 2500,\ \ s_1 = 400 \quad and \quad n_2 = 400,\ \bar{x}_2 = 2200,\ s_2 = 550$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. the sales in two states are equal)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{2500 - 2200}{\sqrt{\dfrac{400^2}{400} + \dfrac{550^2}{400}}} = 8.82$

Taking level of significance as 5%, the table value is $z_{0.05} = 1.96$

Since calculated $|z|$ is greater than the tabulated value z_α, we reject the null hypothesis H_0.

i.e. there is a significant difference in sales of the two cities.

3. **A Mathematics test was given to 50 girls and 75 boys. The girls made an average grade of 76 with an SD of 6 and the boys mad an average grade of 82 with an SD of 2. Test whether there is any difference between the performance of boys and girls.** **(QC 57506 MA 6452 MAY 16)**

Here $n_1 = 50,\ \bar{x}_1 = 76,\ \ s_1 = 6 \quad and \quad n_2 = 75,\ \bar{x}_2 = 82,\ s_2 = 2$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. the performance of boys and girls are equal)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{76 - 75}{\sqrt{\dfrac{6^2}{50} + \dfrac{2^2}{75}}} = 1.137$

Taking level of significance as 5%, the table value is $z_{0.05} = 1.96$

Since calculated $|z|$ is less than the tabulated value z_α, we accept the null hypothesis H_0.

i.e. there is a no difference in the performance of boys and girls.

4. **In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, mean is 15. Could the samples have been drawn from the same population with S.D. 4. Use 1% level of significance.** **(QC 60045 MA 3251 APR 22)**

Here $n_1 = 500,\ \bar{x}_1 = 20,\ \ n_2 = 400,\ \bar{x}_2 = 15,\ \ \sigma = 4$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. the samples have been taken from the same population)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{20 - 15}{4\sqrt{\dfrac{1}{500} + \dfrac{1}{400}}} = 18.6$

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Taking level of significance as 1%, the table value is $z_\alpha = 1.58$

Since calculated $|z|$ is greater than the tabulated value z_α, we conclude that the difference between $\bar{x}_1$ *and* $\bar{x}_2$ is significant at 1% level of significance.

Hence we reject the null hypothesis H_0. i.e. the samples could not have been drawn from the same population.

5. **The mean height of two samples of 1000 and 2000 members are respectively 67.5 and 68 inches. Can they be regarded as drawn from the same population with standard deviation 2.5inches at 5% level of significance? (QC 20753 MA 6452 NOV 18)**

Here $n_1 = 1000, \ \bar{x}_1 = 67.5, \ \ n_2 = 2000, \ \bar{x}_2 = 68, \ \ \sigma = 2.5$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. the samples have been taken from the same population)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{67.5 - 68}{2.5\sqrt{\dfrac{1}{1000} + \dfrac{1}{2000}}} = -5.16$

Taking level of significance as 5%, the table value is $z_{0.05} = 1.96$

Since calculated $|z|$ is greater than the tabulated value z_α, we reject the null hypothesis H_0.

i.e. the samples could not have been drawn from the same population.

6. **A random sample of 100 bulbs from a company P shows a mean life 1300 hours and standard deviation of 82 hrs. Another random sample of 100 bulbs from company Q showed a mean life of 1248 hours and standard deviation of 93 hours. Are the bulbs of company P superior to bulbs of company Q at 5% level of significance. (QC 50782 MA 6452 NOV 17)**

Here $n_1 = 100, \ \bar{x}_1 = 1300, \ \ s_1 = 82 \ \ and \ \ n_2 = 100, \ \bar{x}_2 = 1248, \ \ s_2 = 93$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. both the company bulbs are equally superior)

Alternative Hypothesis: $H_1 = \mu_1 > \mu_2$ (One tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{1300 - 1248}{\sqrt{\dfrac{82^2}{100} + \dfrac{93^2}{100}}} = \dfrac{52}{12.39} = 4.19$

Taking level of significance as 5%, the table value is $z_\alpha = 2.33$

Since calculated $|z|$ is greater than the tabulated value z_α, we reject the null hypothesis H_0.

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i.e. the bulbs of company P is superior to bulbs of company Q.

7. **Given n_1=32, $n_2 = 36$. $\overline{x_1}$ = 72, $\overline{x_2}$ = 74 $s_1 = 8$, $s_2 = 6$**

Test if the means are significant. **(QC 27331 MA 6452 NOV 15)**

Here $n_1 = 32,\ \bar{x}_1 = 72,\ \ s_1 = 8 \quad and \quad n_2 = 36,\ \bar{x}_2 = 74,\ s_2 = 6$

Null Hypothesis $H_0 = \mu_1 = \mu_2$

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{72-74}{\sqrt{\dfrac{8^2}{32} + \dfrac{6^2}{36}}} = -1.15$

Taking level of significance as 5%, the table value is $z_{0.05} = 1.96$

Since calculated $|z|$ is less than the tabulated value z_α, we accept the null hypothesis H_0.

i.e. there is a no difference in the sample means(both come from same population).

8. **The average marks scored by 32 boys is 72 with a SD of 8, while that for 36 girls is 70 with a SD of 6. Test at 1% level of significance whether the boys perform better than girls?**

Here $n_1 = 32,\ \bar{x}_1 = 72,\ \ s_1 = 8 \quad and \quad n_2 = 36,\ \bar{x}_2 = 70,\ s_2 = 6$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (boys and girls perform equally)

Alternative Hypothesis: $H_1 = \mu_1 > \mu_2$ (Right tailed test)

Test statistic $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{72-70}{\sqrt{\dfrac{8^2}{32} + \dfrac{6^2}{36}}} = 1.15$

Taking level of significance as 1%, the table value is $z_\alpha = 2.33$

Since calculated $|z|$ is less than the tabulated value z_α, we accept the null hypothesis H_0.

i.e. we can't conclude that boys perform better than girls.

9. **Given that 32 values obtained for standard wire yielded $\bar{x} = 0.136$ ohm and $s_1 = 0.004$ ohm, and 32 values obtained for alloyed wire yielded $\bar{y} = 0.083$ ohm and $s_2 = 0.005$ ohm. At the 0.05 level of significance, test the claim that the resistance of electric wire can be reduced by more than 0.050 ohm by alloying.** **(QC 20817 MA 8452 APR 22)**

Given n_1=32, $n_2 = 32$. $\overline{x_1} = 0.136$, $\overline{x_2} = 0.083$ $s_1 = 0.004$, $s_2 = 0.005$

1. $H_0 : \mu_1 - \mu_2 = 0.05$ (difference between the resistances of standard wire and alloy wire is 0.05)
2. $H_1 : \mu_1 - \mu_2 > 0.05$ (Right tailed test)
3. Choose level of significance α = 0.05
4. The test statistic $Z = \frac{(\overline{x_1}-\overline{x_2})-0.05}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}} = \frac{0.136-0.083-0.05}{\sqrt{\frac{(0.004)^2}{32}+\frac{(0.005)^2}{32}}} = 2.65$
5. Conclusion: Since calculated value of $|Z| = 2.65 >$ the tabulated value $|z_\alpha| = 1.64$, we reject the null hypothesis H_0.

10. The mean height of 50 male students who showed above average participation in college athletics was 68.2 inches with a standard deviation of 2.5 inches; while 50 male students who showed no interest in such participation had a mean height of 67.5 inches with a standard deviation of 2.8 inches.
a. Test the hypothesis that male students who participate in college athletics are taller than other male students.
b. By how much should the sample size of each of the two groups be increase in order that the observed difference of 0.7 inches in the mean height be significant at the 5% level of significance. **(QC 80610 MA 6452 NOV 16)**

Here $n_1 = 50,\ \overline{x}_1 = 68.2,\ s_1 = 2.5;\quad n_2 = 50,\ \overline{x}_2 = 67.5,\ s_2 = 2.8$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. there is no difference between the means of the population)

Alternative Hypothesis: $H_1 = \mu_1 > \mu_2$ (One tailed test)

Test statistic $z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}} = \frac{68.2-67.5}{\sqrt{\frac{2.5^2}{50}+\frac{2.8^2}{50}}} = 1.32$

Taking level of significance as 5%, the table value is $z_{0.1} = 1.645$

Since calculated $|z|$ is less than the tabulated value z_α, we accept the null hypothesis H_0.

i.e. the height of the male students who participate in college athletics and other male students are same.

To find the sample size if the difference between the two population means are significant.

This may happen if $z \geq 1.645$

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \geq 1.645$$

$$\frac{68.2 - 67.5}{\sqrt{\frac{2.5^2}{n} + \frac{2.8^2}{n}}} \geq 1.645$$

$$\frac{0.7}{\frac{1}{\sqrt{n}}(3.7536)} \geq 1.645$$

$$\sqrt{n} \geq \frac{1.645 \times 3.7536}{0.7}$$

$$n \geq \left[\frac{1.645 \times 3.7536}{0.7}\right]^2$$

$$n \geq 78$$

11. Test the significance of the difference between the means of the samples, drawn from two normal populations with same SD using the following data:

	Size	**Mean**	**SD**
Sample 1	**100**	**61**	**4**
Sample 2	**200**	**63**	**6**

Here $n_1 = 100, \ \bar{x}_1 = 61, \ s_1 = 4; \qquad n_2 = 200, \ \bar{x}_2 = 63, \ s_2 = 6$

Null Hypothesis $H_0 = \mu_1 = \mu_2$ (i.e. there is no difference between the means of the population)

Alternative Hypothesis: $H_1 = \mu_1 \neq \mu_2$ (Two tailed test)

Test statistic $z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}} = \frac{61 - 63}{\sqrt{\frac{4^2}{200} + \frac{6^2}{100}}} = -3.02$

(Note the formula: samples are drawn from two population with same SD)

Taking level of significance as 5%, the table value is $z_{0.05} = 1.96$

Since calculated $|z|$ is greater than the tabulated value z_α, we reject the null hypothesis H_0.

i.e. the populations, from which samples are drawn may not have the same mean.

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EXERCISE

Chi Square Test for Goodness of Fit

1 4 coins were tossed 160 times and the following results were obtained:

No. of Heads	:	0	1	2	3	4
Observed frequencies	:	17	52	54	31	6

Under the assumption that the coins are unbiased, find the expected frequencies of getting 0, 1, 2, 3, 4 heads and test the goodness of fit. (QC 11395 MA 2266 MAY 11)

2 200 digits were chosen at random from a set of tables. The frequencies are given below:

Digit :	0	1	2	3	4	5	6	7	8	9
Frequency:	18	19	23	21	16	25	22	20	21	15

Test the hypothesis that the digits were distributed in equal number in the respective tables.

Chi Square Test for Independence

1 Out of 8000 graduates in a town 800 are females, out of 1600 graduate employees 120 are females. Use χ^2 test to determine if any distinction is made in appointment on the basis of sex. Value of χ^2 at 5% level for one degree of freedom is 3.84. (QC E3126 MA 2266 APR 10)

2 An automobile company gives you the following information about age groups and the liking for particular model of car which it plans to introduce. On the basis of this data, can it be concluded that the model appeal is independent of the age group $\chi^2_{0.05(3)} = 7.815$. (QC E3126 MA 2266 APR 10)

Age group Persons who	Below 20	20 – 39	40 – 59	60 and above
Liked the car	140	80	40	20
Disliked the car	60	50	30	80

3. 1000 students at college level were graded according to their I.Q. and their economic conditions. What conclusion can you drawn from the following data: (QC 21528 MA 2266 MAY 2013)

Economic conditions	IQ Level	
	High	Low
Rich	460	140

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Poor	240	160

Test for Single Mean - Small Sample

1 A sample of 10 boys had the I.Q.'s: 70, 120, 110, 101, 88, 83, 95, 98, 100 and 107. Test whether the population mean I.Q. may be 100. (QC 11491 MA 2266 NOV 2012)

2 The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches? (QC 11395 MA 2266 MAY 2011)

Test for Single Mean - Large Sample

1 The heights of college students in a city are normally distributed with SD 6cm. A sample of 100 students has mean height 158 cms. Test the hypothesis that the mean height of college students in the city is 160cms.

2 The average marks in Mathematics of a sample of 100 students was 51 with a SD of 6 marks. Could this have been a random sample from a population with average marks 50?

Test for Double Mean - Small Sample

1. A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded the following increase in weight.

Diet A :	5	6	8	1	12	4	3	9	6	10
Diet B :	2	3	6	8	10	1	2	8		

Does it show superiority of diet A over diet B?

2. Two independent samples from normal population with equal variance gave the following:

Sample	Size	Mean	SD
1	16	23.4	2.5
2	12	24.9	2.8

Is the difference between the means significant?

3. The following data relate to the marks by 11 students in two tests, one held at the beginning of the year and the other at the end of the year after coaching.

Test 1 :	19	23	16	24	17	18	20	18	21	19	20
Test 2 :	17	24	20	24	20	22	20	20	18	22	19

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Do the data indicate that the students have benefitted by coaching?
(Note: correlated samples)

Test for Double Mean - Large Sample

1. The sales manager of a large company conducted a sample survey in two places A and B taking 200 samples in each case. The results were the following table. Test whether the average sales is the same in the 2 areas at 5% level. (QC 31528 MA 2266 NOV 13)

	Place A	Place B
Average Sales	Rs.2000/-	Rs.1700
SD	Rs.200	Rs.450

2. The sales manager of a large company conducted a sample survey in states A and B taking 400 samples in each case. The results were in the following table. Test whether the average sales is the same in the 2 states at 1% level. (QC 21528 MA 2266 MAY 13)

	State A	State B
Average Sales	Rs.2,500/-	Rs.2,200
SD	Rs.400	Rs.550

Test for Single Proportion

1. 20 people were attacked by a disease and only 18 survived. Will you reject the hypothesis that the survival rate, if attacked by this disease is 85% is favour of the hypothesis that is more at 5% level. (QC 31528 MA 2266 NOV 13)

2. A manufacturer of light bulbs claims that an average of 2% of the bulbs manufactured by him are defective. A random sample of 400 bulbs contained 13 defective bulbs. On the basis of the sample, can you support the manufacturer's claim at 5% level of significance? (QC 51579 MA 2266 MAY 14)

Test for Double Proportions

1. In a random sample of 100 men taken from village A, 60 were found to be consuming alcohol. In another sample of 200 men taken from village B, 100 were found to be consuming alcohol. Do the two villages differ significantly in respect of the proportion of men who consume alcohol? (QC 51579 MA 2266 MAY 14)

2. A machine puts out 16 imperfect articles in a sample of 500. After it was overhauled, it puts out 3 imperfect articles in a sample of 100. Has the machine improved in its performance?

(QC 11491 MA 2266 NOV 2012)

F-Test for Variance

1 Time taken by workers in performing a job are given below: (QC 31528 MA 2266 NOV 13)

Type I :	21	17	27	28	24	23	--
Type II:	28	34	43	36	33	35	39

Test whether there is any significant difference between the variances of time distribution.

2 Test whether there is any significant difference between the variances of the populations from which the following samples are taken: (QC 11491 MA 2266 NOV 2012)

Sample I	:	20	16	26	27	23	22	--
Sample II	:	27	33	42	35	32	34	38

Both t and F-Test

1. Examine whether the difference in the variability in yield is significant at 5% level of significance, for the following: (QC 53192 MA 2266 NOV 2010)

	Set of 40 Plots	Set of 60 Plots
Mean yield per plot	1258	1243
S.D. per plot	34	28

2. Two random samples gave the following data:

Sample	Size	Mean	Variance
I	8	9.6	1.2
II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the same normal population?

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UNIT II – DESIGN OF EXPERIMENTS

Introduction

The design of experiments is a logical construction of the experiment in which the degree of uncertainty with which the inferences is drawn may be well defined. Here we consider some aspects of experimental design and analysis of data from such experiments using ANOVA techniques.

Statistical experiment is conducted to verify the truthiness of a hypothesis. Consider an agricultural experiment that a particular manure increases the yield of a grain. Here the quantity of manure used and quantity of yield are two experimental variables. In addition, there are other variables such as nature of soil, proper watering and quality of seeds also affect the yield, which are called extraneous variables.

So the main aim of our design of experiment is to control the extraneous variables and hence to minimize the experimental error so that the results of the experiments could be attributed only to the experimental variables.

What is the aim of the design experiment? MA 6452 NOV 17

The purpose of experimental design is to obtain maximum information with the minimum cost and labour.

With respect to an agricultural experiment, we mean the factors used in this design like treatments, experimental unit, blocks and experimental error as follows:

Treatments: Types of crops, variety of manure, methods of cultivation
Experimental Unit: The plot of land
Blocks: Division of the land separated which are relatively homogeneous divisions
Error: Variation in the yield due to extraneous variables.

Basic Principles of Experimental Design

The basic principles of experimental design are (i) randomization (ii) replication and (iii) local control and (iv) ANOVA.

What are the basic principles of experimental design?

MA3251 APR 22

Randomization controls the effect of extraneous variables. It is done by the selection of plots for experimental groups and control groups in a random manner.

Replication means repetition. In our example, the manure is used in more than one plot so that the effect may be identified precisely.

Local control controls the effect of extraneous variable by using the methods such as grouping, blocking and balancing.

ANOVA is a test of the homogeneity of a set of data. It is defined as The separation of the variance ascribable to one group of causes from the variance ascribable to other groups.

Write short notes on Analysis of Variance. MA3251 APR 22

https://doi.org/10.5281/zenodo.15289873

It enables us to find the total variability due to each factor and by comparing these variation, homogeneity of the observation may be tested. i.e. whether all the observations are drawn from the same normal population.

What are the uses of ANOVA? MA 6452 MAY 17

Assumptions for ANOVA test

What are the basic assumptions involved in ANOVA?

- The individual samples are drawn randomly from the population
- The variance between the samples is constant
- The sampled population is normal
- Experimental errors should be homogeneous and are independent.

Experimental Error

Define experimental error. MA 6452 NOV 16

The unexplained random part of the variation in any experiment is termed as experimental error. An estimate of experimental error can be obtained by replication.

Basic Designs of Experimental Design

The basic designs of experiment are
(i) One way classification (Completely Randomized Design)
(ii) Two way classification (Randomized Block Design)
(iii) Three way classification (Latin Square Design)

What are the basic designs of Experiment? MA 6452 MAY 18

Completely Randomized Design

Explain CRD with an example. MA 8452 APR 22

The term CRD or one way classification refers to the fact that a single variable factor of interest is controlled and its effect on the other elementary units is observed. Suppose we wish to compare h treatments (say manure) and there are n plots available for the experiment. Let i^{th} treatment be replicated n_i times, so that $n_1 + n_2 + ... + n_h = n$.

In this design treatments are randomly arranged over the experimental units which are divided into groups at random as follows.

The plots are numbered from 1 to n serially. n identical cards are taken, numbered from 1 to n and shuffled thoroughly. The numbers on the first n_1 cards drawn randomly give the number of plots to which the first treatment is to be given. The numbers on the next n_2 cards drawn at random give the numbers of the plots to which the second treatment is to be given and so on.

This design is called a Completely Randomized Design. This design is used only when the number of treatments is small and the experimental material is homogeneous.

For example, in on each of the several blocks agricultural field experiments, where several varieties of wheat are to be tested on each of several blocks of land, it is necessary to assign the varieties at random to several plots in each block.

https://doi.org/10.5281/zenodo.15289873

One Way Classification

Here the data are classified on the basis of one criterion as follows

Treatment	Values
1	x_{11} x_{12} ...... x_{1i} ..... x_{1n_1}
2	x_{21} x_{22} ...... x_{2i} ..... x_{2n_2}
:	:
:	:
k	x_{k1} x_{k2} ...... x_{ki} ..... x_{kn_k}

Then $\sum_{i=1}^{k} n_i = N$

Here we wish to test the null hypothesis that there is no significant difference between the treatments under consideration. i.e. $H_0 : \mu_1 = \mu_2 = ... = \mu_k$ and hence the alternative hypothesis is $H_1 : \mu_1 \neq \mu_2 \neq ..... \neq \mu_k$

Computational formula for various sum of squares:

Total sum of square $V = \sum \sum x_{ij}^2 - \frac{T^2}{N}$ where $T = \sum \sum x_{ij}$

Sum of squares between samples $V_1 = \sum \left(\frac{T_i}{n_i}\right)^2 - \frac{T^2}{N}$

Sum of squares within samples(Error) $V_2 = V - V_1$

ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F
Treatment	V_1	$k-1$	$\frac{SST}{k-1} = S_T^2$	$F = \frac{S_T^2}{S_E^2}$ $\left(S_T^2 > S_E^2\right)$
Error	V_2	$N-k$	$\frac{SSE}{N-k} = S_E^2$	
Total	V	$N-1$		

Here the calculated ratio follows F distribution with degrees freedom $(k-1, N-k)$. If the calculated value of F is less than the tabulated value, then the null hypothesis is accepted. Otherwise it is rejected.

1. **Write two advantages of completely randomized experimental design.**
(QC 27331 MA 6452 NOV 15)

- CRD results in the maximum use of the experimental units since all the experimental materials can be used.
- The design is very flexible and easy to layout
- Any number of replicates and treatments may be used
- It provides with the maximum number of degrees of freedom
- It is most useful for laboratory techniques and methodological studies

2. **What are the basic elements of an ANOVA table for one way classification?**
(QC 41313 MA 6452 MAY 18)

ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F
Treatment	SST	$k-1$	$\frac{SST}{k-1} = S_T^2$	$F = \frac{S_T^2}{S_E^2}$ $\left(S_T^2 > S_E^2\right)$
Error	SSE	$N-k$	$\frac{SSE}{N-k} = S_E^2$	
Total		$N-1$		

Solved Problems

3. **The following table gives the yields of 15 samples of plot under three varieties of seed.**

A	**20**	**21**	**23**	**16**	**20**
B	**18**	**20**	**17**	**15**	**25**
C	**25**	**28**	**22**	**28**	**32**

Test using analysis of variance whether there is a significant difference in the average yield of seeds. **(QC 80610 MA 6452 NOV 16)**

This is one way classification. Let us tabulate the data:

Varieties of seeds	Plots					Total	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
	1	2	3	4	5						
A	20	21	23	16	20	100	400	441	529	256	400
B	18	20	17	15	25	95	324	400	289	225	625
C	25	28	22	28	32	103	625	784	484	784	1024
Total						298	1349	1625	1302	1265	2049

https://doi.org/10.5281/zenodo.15289873

H_0 : There is no difference between the varieties of seeds in respect of growth.

H_1 : There is significant difference between the varieties of seeds in respect of growth.

Step 1 : Number of data N = 15

Step 2. Total T = 298

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(298)^2}{15} = 5920.266$

Step 4. Total Sum of Squares $V = \sum x_1^2 + \sum x_2^2 + \sum x_3^2 + \sum x_4^2 + \sum x_5^2 - \frac{T^2}{N}$

$= 1349 + 1625 + 1302 + 1265 + 2049 - 5920.266$

$= 1669.73$

Step 5. Sum of Squares between varieties of seeds $V_1 = \frac{\left(\sum T_1\right)^2}{n_1} + \frac{\left(\sum T_2\right)^2}{n_2} + \frac{\left(\sum T_3\right)^2}{n_3} - \frac{T^2}{N}$

$$V_1 = \frac{(100)^2}{5} + \frac{(95)^2}{5} + \frac{(103)^2}{5} - 5920.266$$

$$= 6.534$$

Step 6. Sum of Squares within varieties of seeds $V_2 = V - V_1 = 1669.73 - 6.534 = 1663.19$

Step 7. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between varieties of seeds	6.534	$k-1$ 3 – 1 = 2	$\frac{6.534}{2} = 3.267$	$\frac{138.59}{3.267} = 42.42$	$F_{0.05}(12,2)$ $= 19.41$
Within varieties of seeds	1663.19	$N-k$ 15 – 3 = 12	$\frac{1663.19}{12} = 138.59$		
Total	1669.724	$N-1=14$			

Step 7 : Conclusion : Here calculated value is greater than the tabulated value.

https://doi.org/10.5281/zenodo.15289873

Therefore , Null hypothesis is rejected. i.e. There is significant difference between the varieties of seeds in respect of growth.

4. The accompanying data resulted from an experiment comparing the degree of soiling for fabric copolymerized with the 3 different mixtures of methacrylic acid. Analyse the classification.

Mixture1	**0.56**	**1.12**	**0.90**	**1.07**	**0.94**
Mixture 2	**0.72**	**0.69**	**0.87**	**0.78**	**0.91**
Mixture 3	**0.62**	**1.08**	**1.07**	**0.99**	**0.93**

(QC 72071 MA 6452 MAY 17)

This is one way classification. Let us tabulate the data:

Mixture	Degree of Soiling					Total	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
	1	2	3	4	5						
M1	0.56	1.12	0.9	1.07	0.94	4.59	0.314	1.254	0.81	1.145	0.884
M2	0.72	0.69	0.87	0.78	0.91	3.97	0.518	0.476	0.757	0.608	0.828
M3	0.62	1.08	1.07	0.99	0.93	4.69	0.384	0.384	1.145	0.98	0.865
Total						13.25	1.216	2.114	2.712	2.733	2.577

H_0: There is no difference between the degree of soiling with respect to the mixtures

H_1: There is significant difference between the degree of soiling with respect to the mixtures

Step 1 : Number of data N = 15

Step 2. Total T = 13.25

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(13.25)^2}{15} = 11.704$

Step 4. Total Sum of Squares $V = \sum x_1^2 + \sum x_2^2 + \sum x_3^2 + \sum x_4^2 + \sum x_5^2 - \frac{T^2}{N}$

$= 1.216 + 2.114 + 2.712 + 2.733 + 2.577 - 11.704$

$= 12.1351 - 11.704$

$= 0.4311$

Step 5. Sum of Squares between degree of soiling $V_1 = \frac{\left(\sum T_1\right)^2}{n_1} + \frac{\left(\sum T_2\right)^2}{n_2} + \frac{\left(\sum T_3\right)^2}{n_3} - \frac{T^2}{N}$

$$V_1 = \frac{4.59^2}{5} + \frac{3.97^2}{5} + \frac{4.69^2}{5} - 11.704$$

$$= 0.061$$

https://doi.org/10.5281/zenodo.15289873

Step 6. Sum of Squares within degree of soiling $V_2 = V - V_1 = 0.4311 - 0.061 = 0.3701$

Step 7. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between degree of soiling	0.061	$k-1$ 3 – 1 = 2	$\frac{0.061}{2} = 0.0305$	$\frac{0.0308}{0.0305} = 1.011$	$F_{0.05}(12, 2)$ $= 19.41$
Within degree of soiling	0.3701	$N-k$ 15 – 3 = 12	$\frac{0.3701}{12} = 0.0308$		
Total	36	$N-1=11$			

Step 7 : Conclusion : Here calculated value is less than tabulated value.
Therefore , Null hypothesis is accepted. i.e. There is no difference between the degree of soiling with respect to the mixtures

5. A set of data involving 4 tropical food stuffs A, B, C, D tried on 20 chicks is given below. All the 20 chicks are treated alike in all respects except the feeding treatments and each feeding treatment is given to 5 chicks. Analyse the data:

A	**55**	**49**	**42**	**21**	**52**
B	**61**	**112**	**30**	**89**	**63**
C	**42**	**97**	**81**	**95**	**92**
D	**169**	**137**	**169**	**85**	**154**

(QC 72071 MA 6452 MAY 17)

This is one way classification. Let us tabulate the data by taking 75 as origin:

Food Stuffs	Growth of chicks					Total	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
	1	2	3	4	5						
A	–20	–26	–33	–54	–23	–156	400	676	1089	2916	529
B	–14	37	–45	14	–12	–20	196	1369	2025	196	144
C	–33	22	6	20	17	32	1089	484	36	200	289
D	94	62	94	10	79	339	8836	3844	8836	100	6241
Total						195	10521	6373	11986	3412	7203

H_0 : There is no difference between the tropical food stuffs with respect to the growth of chicks.

H_1 : There is significant difference between the tropical food stuffs with respect to the growth of chicks.

https://doi.org/10.5281/zenodo.15289873

Step 1 : Number of data N = 20

Step 2. Total T = 195

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(195)^2}{20} = 1901.25$

Step 4. Total Sum of Squares $V = \sum x_1^2 + \sum x_2^2 + \sum x_3^2 + \sum x_4^2 - \frac{T^2}{N}$

= 10521 + 6373 + 11986 +3412 + 7203 – 1901.25

= 37593.75

Step 5. Sum of Squares between Food Stuffs $V_1 = \frac{\left(\sum T_1\right)^2}{n_1} + \frac{\left(\sum T_2\right)^2}{n_2} + \frac{\left(\sum T_3\right)^2}{n_3} - \frac{T^2}{N}$

$$V_1 = \frac{(-156)^2}{5} + \frac{(-20)^2}{5} + \frac{32^2}{5} + \frac{339^2}{5} - 1901.25$$

$$= 26234.95$$

Step 6. Sum of Squares within Food Stuffs $V_2 = V - V_1 = 37593.75 - 26234.95 = 11358.8$

Step 7. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Food Stuffs	26234.95	$k-1$ 4 – 1 = 3	$\frac{26234.95}{3}$ $= 8744.98$	$\frac{8744.98}{709.925} = 12.31$	$F_{0.05}(3,16)$ $= 3.24$
Within Food Stuffs	11358.8	$N-k$ 20 – 4 = 16	$\frac{11358.8}{16}$ $= 709.925$		
Total	37593.75	$N-1=19$			

Step 7 : Conclusion : Here calculated value is greater than the tabulated value. Therefore , Null hypothesis is rejected. i.e. There is significant difference between the tropical food stuffs with respect to the growth of chicks

Randomised Block Design

Consider an agricultural experiment using which we wish to test the effect of k fertilizing treatments on the yield of a crop. We assume that soil fertility of the plots are known.

Define Randomized Block Design.

https://doi.org/10.5281/zenodo.15289873

Then we divide the plots into h blocks, according to the soil fertility, each block containing k plots. Thus the plots in each block will be of homogeneous fertility.

MA 8452 APR 22

Within each block, the k treatments are given to the k plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same k treatments are repeated from block to block. This design is called Randomized Block Design.

Advantages of RBD

This is more accurate than completely randomized design

Any number of treatments on the number of replicates may be used

Statistical analysis is simple and fast.

State about advantages of a randomized block experimental design.
MA 6452 MAY 19

Note: It is not suitable (i) for large number of treatments (ii) if blocks are not homogeneous

Comparison of RBD and CRD

RBD is more efficient than CRD

Experimental error of RBD is very less than CRD

In RBD, treatments are allocated at random within the units of each stratum, but it is not done in CRD.

RBD is more flexible than CRD because there is no restrictions on the number of treatments or replications

Two Way Classification

The data collected from experiments with RBD form a two way classification i.e. classified according to two factors say blocks (r) and treatments (k).

Here the data are classified on the basis of one criterion as follows

		Treatments
		1 2 .... k
Blocks	1	x_{11} x_{12} ...... x_{1i} ..... x_{1k}
	2	x_{21} x_{22} ...... x_{2i} ..... x_{2k}
	:	:
	:	:
	r	x_{r1} x_{r2} ...... x_{ri} ..... x_{rk}

Then $rk = N$

Here we wish to test the null hypothesis that there is no significant difference between the treatments as well as blocks under consideration. i.e.

$H_{01} : \mu_1 = \mu_2 = ... = \mu_r$ and $H_{02} : \mu_1 = \mu_2 = ... = \mu_k$

Computational formula for various sum of squares:

Total sum of square $V = \sum \sum x_{ij}^2 - \frac{T^2}{N}$ where $T = \sum \sum x_{ij}$

Sum of squares between blocks $V_1 = \sum \left(\frac{T_i}{r}\right)^2 - \frac{T^2}{N}$

Sum of squares between treatments $V_2 = \sum \left(\frac{T_j}{k}\right)^2 - \frac{T^2}{N}$

Sum of squares within samples(Error) $V_2 = V - V_1 - V_2$

ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F
Blocks	V_1	$r-1$	$\frac{V_1}{r-1} = S_R^2$	$F_1 = \frac{S_R^2}{S_E^2} > 1$
Treatments	V_2	$k-1$	$\frac{V_2}{k-1} = S_C^2$	$F_2 = \frac{S_C^2}{S_E^2} > 1$
Error	V_3	$(r-1)(k-1)$	$\frac{V_3}{(r-1)(k-1)} = S_E^2$	
Total	V	$N-1$		

Here the calculated ratios F_1 , F_2 follows F distribution with degrees freedom $(r-1,(r-1)(k-1))$ and $(k-1,(r-1)(k-1))$ respectively.

If the calculated value of F is less than the tabulated value, then the null hypothesis is accepted. Otherwise it is rejected.

Solved Problems

https://doi.org/10.5281/zenodo.15289873

1. Performa 2-way ANOVA on the data given below:

	Treatment-I 1	**2**	**3**
1	**30**	**26**	**38**
2	**24**	**29**	**28**
Treatment-II3	**33**	**24**	**35**
4	**36**	**31**	**30**
5	**27**	**35**	**33**

Use the coding method subtracting 30 from the given number. **(QC 72071 MA 6452 MAY 17)**

This is two way classification. Calculation table. Subtract 30 from all the values.

	Treatment-I 1	2	3	Row Total T_R	x_1^2	x_2^2	x_3^2
Treatment-II 1	0	–4	8	4	0	16	64
2	–6	–1	–2	–9	36	1	4
3	3	–6	5	2	9	36	25
4	6	1	0	7	36	1	0
5	–3	5	3	5	9	25	9
Column Total T_C	0	–5	14	**9**	90	79	102
					$\sum x_{ij}^2 = 271$		

H_{01}: There is no difference between treatment-II with respect to wellness.

H_{02}: There is no difference between treatment-I with respect to wellness.

Step 1 : Number of data N = 15

Step 2. Total T = 9

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(9)^2}{15} = 5.4$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 271 - 5.4 = 265.6$

https://doi.org/10.5281/zenodo.15289873

Step 5. Sum of Squares between treatment-II

$$V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} + \frac{\left(\sum T_{R4}\right)^2}{n_4} + \frac{\left(\sum T_{R5}\right)^2}{n_5} - \frac{T^2}{N}$$

$$V_1 = \frac{(4)^2}{3} + \frac{(-9)^2}{3} + \frac{2^2}{3} + \frac{7^2}{3} + \frac{5^2}{3} - 5.4$$

$$= 52.9$$

Step 6. Sum of Squares between treatment-I $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} - \frac{T^2}{N}$

$$V_2 = \frac{0^2}{5} + \frac{(-5)^2}{5} + \frac{(14)^2}{5} - 5.4$$

$$= 38.8$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 265.6 - 52.9 - 38.8 = 173.9$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Treatment-II	52.9	$m-1$ 5 – 1 = 4	$\frac{52.9}{4} = 13.22$	$\frac{21.7}{13.22} = 1.64$	$F_{0.05}(8, 4) = 6.04$
Between Treatment-I	38.8	$n-1$ 3 – 1 = 2	$\frac{38.8}{2} = 19.4$	$\frac{21.7}{19.4} = 1.11$	$F_{0.05}(8, 2) = 19.37$
Error	173.9	$(m-1)(n-1)$ = 8	$\frac{173.9}{8} = 21.7$		
Total	24.6	$N-1=14$			

Step 7 : Considering the difference between treatment-II, we find that, calculated value of $F = 1.64 <$ tabulated value of $F_{5\%} = 6.04$, we accept H_{01} : (the treatment-II do not differ significantly)

Considering the difference between treatment-I, we find that, calculated value of $F = 1.11 <$ tabulated value of $F_{5\%} = 19.37$, we accept H_{02} : (the treatment-I do not differ significantly)

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2. **Three varieties of coal were analysed by 4 chemists and the ash content is tabulated here. Perform an analysis of variance.**

	Chemists A	B	C	D
Coal I	**8**	**5**	**5**	**7**
Coal II	**7**	**6**	**4**	**4**
Coal III	**3**	**6**	**5**	**4**

(QC 57506 MA 6452 MAY 16)

This is two way classification. Calculation table.

		Chemists				Row Total T_R	x_1^2	x_2^2	x_3^2	x_4^2
		A	B	C	D					
Coal	I	8	5	5	7	25	64	25	25	49
	II	7	6	4	4	21	49	36	16	16
	III	3	6	5	4	18	9	36	25	16
Column Total T_C		18	17	14	15	64	122	97	66	81
							$\sum x_{ij}^2 = 366$			

H_{01} : There is no difference between coal with respect to ash content.

H_{02} : There is no difference between chemists with respect to ash content.

Step 1 : Number of data N = 12

Step 2. Total T = 64

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(64)^2}{12} = 341.3$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 366 - 341.3 = 24.6$

Step 5. Sum of Squares between coals $V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} - \frac{T^2}{N}$

https://doi.org/10.5281/zenodo.15289873

$$V_1 = \frac{(25)^2}{4} + \frac{21^2}{4} + \frac{18^2}{4} - 341.3$$

$$= 6.2$$

Step 6. Sum of Squares between chemists $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} + \frac{\left(\sum T_{C4}\right)^2}{n_4} - \frac{T^2}{N}$

$$V_2 = \frac{18^2}{3} + \frac{17^2}{3} + \frac{(14)^2}{3} + \frac{(15)^2}{3} - 341.3$$

$$= 3.36$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 24.6 - 6.2 - 3.36 = 15.04$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Coals	6.2	$m-1$ 3 – 1 = 2	$\frac{6.2}{2} = 3.1$	$\frac{3.1}{2.5} = 1.24$	$F_{0.05}(2, 6) = 5.14$
Between Chemists	3.36	$n-1$ 4 – 1 = 3	$\frac{3.36}{3} = 1.12$	$\frac{2.5}{1.12} = 2.2$	$F_{0.05}(6,3) = 8.94$
Error	15.04	$(m-1)(n-1)$ = 6	$\frac{15.04}{6} = 2.5$		
Total	24.6	$N-1=11$			

Step 7 : Considering the difference between Coals, we find that, calculated value of $F = 1.24 <$ tabulated value of $F_{5\%} = 5.14$, we accept H_{01} : (the coals do not differ significantly)

Considering the difference between chemists, we find that, calculated value of $F = 2.2 <$ tabulated value of $F_{5\%} = 8.94$, we accept H_{02} : (the chemists do not differ significantly)

3. The following data represent a certain person to work from Monday to Friday by four different routes.

	Days				
	Mon	**Tue**	**Wed**	**Thu**	**Fri**

https://doi.org/10.5281/zenodo.15289873

	1	**22**	**26**	**25**	**25**	**31**
Routes	**2**	**25**	**27**	**28**	**26**	**29**
	3	**26**	**29**	**33**	**30**	**33**
	4	**26**	**28**	**27**	**30**	**30**

Test at 5% level of significance whether the differences among the means obtained for the different routes are significant and also whether the differences among the means obtained for the different days of the week are significant. **(QC 20753 MA 6452 NOV 18)**

This is two way classification. Let us arrange the data by subtracting 26 from each value.

		Days					Row Total T_R	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
		1	2	3	4	5						
Routs	1	–4	0	–1	–1	5	–1	16	0	1	1	25
	2	–1	1	2	0	3	5	1	1	4	0	9
	3	0	3	7	4	7	21	0	9	49	16	49
	4	0	2	1	4	4	11	0	4	1	16	16
Column Total T_C		–5	6	9	7	19	**36**	17	14	55	33	90
								$\sum x_{ij}^2 = 209$				

H_{01} : There is no difference between routes with respect to work.

H_{02} : There is no difference between days with respect to work.

Step 1 : Number of data N = 20

Step 2. Total T = 36

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(36)^2}{20} = 64.8$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 209 - 64.8 = 144.2$

Step 5. Sum of Squares between routes $V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} + \frac{\left(\sum T_{R4}\right)^2}{n_4} - \frac{T^2}{N}$

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$$V_1 = \frac{-1^2}{5} + \frac{5^2}{5} + \frac{21^2}{5} + \frac{11^2}{5} - 64.8$$

$$= 52.8$$

Step 6. Sum of Squares between days $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} + \frac{\left(\sum T_{C4}\right)^2}{n_4} + \frac{\left(\sum T_{C5}\right)^2}{n_5} - \frac{T^2}{N}$

$$V_2 = \frac{-5^2}{4} + \frac{6^2}{4} + \frac{9^2}{4} + \frac{7^2}{4} + \frac{19^2}{4} - 64.8$$

$$= 73.2$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 144.2 - 52.8 - 73.2 = 18.2$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Routes	52.8	$m-1$ 4 – 1 = 3	$\frac{52.8}{3} = 17.6$	$\frac{17.6}{1.52} = 11.57$	$F_{0.05}(3,12) = 3.49$
Between Days	73.2	$n-1$ 5 – 1 = 4	$\frac{73.2}{4} = 18.3$	$\frac{18.53}{1.52} = 12.2$	$F_{0.05}(3,12) = 3.49$
Error	18.2	$(m-1)(n-1)$ = 12	$\frac{18.2}{12} = 1.52$		
Total	144.2	$N-1=19$			

Step 7 : Considering the difference between Routes, we find that, calculated value of $F = 11.57 >$ tabulated value of $F_{5\%} = 3.49$, we reject H_{01} : (the routes differ significantly)

Considering the difference between Days, we find that, calculated value of $F = 12.2 >$ tabulated value of $F_{5\%} = 3.49$, we reject H_{02} : (the Days differ significantly)

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4. The following table gives the number of refrigerators sold by 4 salesman in 3 months May, June, July.

Month	**Salesman**			
May	**50**	**40**	**48**	**39**
June	**46**	**48**	**50**	**45**
July	**39**	**44**	**40**	**39**

Is this a significant difference in the sales made by 4 salesman?
Is this a significant difference in the sales during different month?(QC 53250 MA 6452 MAY 19)

This is two way classification. Let us arrange the data by subtracting 40 from each value.

		Salesman				Row Total T_R	x_1^2	x_2^2	x_3^2	x_4^2
		1	2	3	4					
Month	May	10	0	8	–1	17	100	0	64	1
	June	6	8	10	5	29	36	64	100	25
	July	–1	4	0	–1	2	1	16	0	1
Column Total T_C		15	12	18	3	48	137	80	164	27
							$\sum x_{ij}^2 = 408$			

H_{01} : There is no difference between salesmen with respect to sales.

H_{02} : There is no difference between month with respect to sales.

Step 1 : Number of data N = 12

Step 2. Total T = 48

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(48)^2}{12} = 192$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 408 - 192 = 216$

Step 5. Sum of Squares between months $V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} - \frac{T^2}{N}$

$$V_1 = \frac{17^2}{4} + \frac{29^2}{4} + \frac{2^2}{4} - 216$$

$$= 74.5$$

Step 6. Sum of Squares between salesman $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} + \frac{\left(\sum T_{C4}\right)^2}{n_4} - \frac{T^2}{N}$

$$V_2 = \frac{15^2}{3} + \frac{12^2}{3} + \frac{18^2}{3} + \frac{3^2}{3} - 216$$

$$= 18$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 216 - 74.5 - 18 = 123.5$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Months	74.5	$m-1$ 3 – 1 = 2	$\frac{74.5}{2} = 37.25$	$\frac{37.25}{20.58} = 1.81$	$F_{0.05}(2,6) = 5.14$
Between Salesman	18	$n-1$ 4 – 1 = 3	$\frac{18}{3} = 6$	$\frac{20.58}{6} = 3.43$	$F_{0.05}(6,3) = 8.94$
Error	123.5	$(m-1)(n-1)$ = 6	$\frac{123.5}{6} = 20.58$		
Total	216	$N-1=11$			

Step 7 : Considering the difference between Months, we find that, calculated value of $F = 1.81 <$ tabulated value of $F_{5\%} = 5.14$, we accept H_{01} : (the months do not differ significantly)

Considering the difference between Salesman, we find that, calculated value of $F = 3.43 <$ tabulated value of $F_{5\%} = 8.94$, we accept H_{02} : (the salesman do not differ significantly)

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5. Perform ANOVA and test at 0.05 level of significance whether these are differences in the detergents or in the engines for the given data.

		Engine	
	A	**B**	**C**
Detergent I	**45**	**31**	**51**
II	**47**	**46**	**52**
III	**48**	**50**	**55**
IV	**42**	**37**	**49**

(QC 27331 MA 6452 NOV 15)

This is two way classification. Let us arrange the data by subtracting 40 from each value.

		Engine			Row Total T_R	x_1^2	x_2^2	x_3^2
		A	B	C				
Detergent	I	5	–9	11	7	25	81	121
	II	7	6	12	25	49	36	144
	III	8	10	15	33	64	100	225
	IV	2	–3	9	8	4	9	81
Column Total T_C		22	4	47	**73**	142	226	571
						$\sum x_{ij}^2 = 939$		

H_{01} : There is no difference between detergents with respect to washing.

H_{02} : There is no difference between engine with respect to washing.

Step 1 : Number of data N = 12

Step 2. Total T = 73

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(73)^2}{12} = 444.1$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 939 - 444.1 = 494.9$

Step 5. Sum of Squares between detergents $V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} + \frac{\left(\sum T_{R4}\right)^2}{n_4} - \frac{T^2}{N}$

$$V_1 = \frac{7^2}{3} + \frac{25^2}{3} + \frac{33^2}{3} + \frac{8^2}{3} - 444.1$$

$$= 164.9$$

Step 6. Sum of Squares between engines $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} - \frac{T^2}{N}$

$$V_2 = \frac{22^2}{4} + \frac{4^2}{4} + \frac{47^2}{4} - 444.1$$

$$= 233.15$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 494.9 - 164.9 - 233.15 = 96.85$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Detergents	164.9	$m-1$ 4 – 1 = 3	$\frac{164.9}{3} = 54.96$	$\frac{54.96}{16.14} = 3.4$	$F_{0.05}(3,6) = 4.76$
Between Engines	233.15	$n-1$ 3 – 1 = 2	$\frac{233.15}{2} = 116.5$	$\frac{116.5}{16.14} = 7.2$	$F_{0.05}(2, 6) = 5.14$
Error	96.85	$(m-1)(n-1)$ = 6	$\frac{96.85}{6} = 16.14$		
Total	494.9	$N-1=11$			

Step 7 : Considering the difference between Detergents, we find that, calculated value of $F = 3.4 <$ tabulated value of $F_{5\%} = 4.76$.

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Therefore we accept H_{01} : (the detergents do not differ significantly)

Considering the difference between Engines, we find that, calculated value of $F = 7.2 >$ tabulated value of $F_{5\%} = 5.14$.

Therefore, we reject H_{02} : (the Engines differ significantly)

6. A company appoints 4 salesmen A, B, C and D and observes their sales in 3 seasons, summer, winter and monsoon. The figures are given in the following table:

Season	**Salesman**			
	A	**B**	**C**	**D**
Summer	**45**	**40**	**28**	**37**
Winter	**43**	**41**	**45**	**38**
Monsoon	**39**	**39**	**43**	**41**

Carry out an analysis of variance. **(QC 80610 MA 6452 NOV 16)**

This is two way classification. Let us arrange the data by subtracting 40 from each value.

		Salesman				Row Total T_R	x_1^2	x_2^2	x_3^2	x_4^2
		A	B	C	D					
Season	Summer	5	0	–12	–3	–10	25	0	144	9
	Winter	3	1	5	–2	7	9	1	25	4
	Monsoon	–1	–1	3	1	2	1	1	9	1
Column Total T_C		7	0	–4	–4	–1	35	2	164	14
							$\sum x_{ij}^2 = 215$			

H_{01} : There is no difference between salesmen with respect to sales.

H_{02} : There is no difference between season with respect to sales.

Step 1 : Number of data N = 12

Step 2. Total T = –1

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Step 3. Correction Factor $\frac{\mathrm{T}^2}{\mathrm{N}} = \frac{(-1)^2}{12} = 0.083$

Step 4. Total Sum of Squares $V = \sum x_{ij}^2 - \frac{T^2}{N} = 215 - 0.083 = 214.9$

Step 5. Sum of Squares between seasons $V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} - \frac{T^2}{N}$

$$V_1 = \frac{(-10)^2}{4} + \frac{7^2}{4} + \frac{2^2}{4} - 0.083$$

$$= 38.16$$

Step 6. Sum of Squares between salesman $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} + \frac{\left(\sum T_{C4}\right)^2}{n_4} - \frac{T^2}{N}$

$$V_2 = \frac{7^2}{3} + \frac{0^2}{3} + \frac{(-4)^2}{3} + \frac{(-4)^2}{3} - 0.083$$

$$= 26.9$$

Step 7. Error Sum of Squares $V_3 = V - V_1 - V_2 = 214.9 - 38.16 - 26.9 = 149.84$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 5% level
Between Season	38.16	$m-1$ 3 – 1 = 2	$\frac{38.16}{2} = 19.08$	$\frac{24.9}{19.48} = 1.27$	$F_{0.05}(6,2) = 19.33$
Between Salesman	26.9	$n-1$ 4 – 1 = 3	$\frac{26.9}{3} = 8.96$	$\frac{24.9}{8.96} = 2.77$	$F_{0.05}(6,3) = 8.94$

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Error	149.84	$(m-1)(n-1)$ = 6	$\frac{149.84}{6} = 24.9$		
Total	214.9	$N-1=11$			

Step 7 : Considering the difference between Seasons, we find that, calculated value of $F = 1.27 <$ tabulated value of $F_{5\%} = 19.33$, we accept H_{02} : (the seasons do not differ significantly)

Considering the difference between Salesman, we find that, calculated value of $F = 2.77 <$ tabulated value of $F_{5\%} = 8.94$, we accept H_{01} : (the salesman do not differ significantly)

7. A chemist wishes to test the effect of four chemical agents on the strength of a particular type of cloth. Because there might be variability from one bolt to another, the chemist decides to use a randomized block design, with the bolts of cloth consider as blocks, she selects five bolts and applies all four chemicals in random order to each bolt. The resulting tensile strengths follows:

		Bolt				
		1	**2**	**3**	**4**	**5**
	1	**73**	**68**	**74**	**71**	**67**
Chemical	**2**	**73**	**67**	**75**	**72**	**70**
	3	**75**	**68**	**78**	**73**	**68**
	4	**73**	**71**	**75**	**75**	**69**

Does the tensile strength depend on chemical? Test at $\alpha = 0.10$. (QC 41313 MA 6452 MAY 18)

Since the design is RBD, this is two way classification. Let us tabulate the data by taking 71 as origin:

Chemical	Bolts of cloths					Row Total T_R	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2
	1	2	3	4	5						
1	2	–3	3	0	–4	–2	4	9	9	0	16
2	2	–4	4	1	–1	2	4	16	16	1	1
3	4	–3	7	2	–3	7	16	9	49	4	9
4	2	0	4	4	–2	8	4	0	16	16	4
Column Total T_C	10	–10	18	7	–10	**15**	28	34	90	21	30

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H_{01} : There is no difference between the chemicals with respect to the tensile strength.

H_{02} : There is no significant difference between the bolts of cloth with respect to the tensile strength..

Step 1 : Number of data N = 20

Step 2. Total T = 15

Step 3. Correction Factor $\frac{T^2}{N} = \frac{(15)^2}{20} = 11.25$

Step 4. Total Sum of Squares $V = \sum x_1^2 + \sum x_2^2 + \sum x_3^2 + \sum x_4^2 - \frac{T^2}{N}$

= 28 + 34 + 90 + 21 + 30 – 11.25

= 191.75

Step 5. Sum of Squares between chemicals

$$V_1 = \frac{\left(\sum T_{R1}\right)^2}{n_1} + \frac{\left(\sum T_{R2}\right)^2}{n_2} + \frac{\left(\sum T_{R3}\right)^2}{n_3} + \frac{\left(\sum T_{R4}\right)^2}{n_4} + \frac{\left(\sum T_{R5}\right)^2}{n_5} - \frac{T^2}{N}$$

$$V_1 = \frac{(-2)^2}{5} + \frac{2^2}{5} + \frac{7^2}{5} + \frac{8^2}{5} - 11.25$$

$$= 12.95$$

Step 6. Sum of Squares between Bolts $V_2 = \frac{\left(\sum T_{C1}\right)^2}{n_1} + \frac{\left(\sum T_{C2}\right)^2}{n_2} + \frac{\left(\sum T_{C3}\right)^2}{n_3} + \frac{\left(\sum T_{C4}\right)^2}{n_4} - \frac{T^2}{N}$

$$V_2 = \frac{10^2}{4} + \frac{(-10)^2}{4} + \frac{(18)^2}{4} + \frac{(7)^2}{4} + \frac{(-10)^2}{4} - 11.25$$

$$= 157$$

Step 7. Sum of Squares within Chemicals $V_3 = V - V_1 - V_2 = 191.75 - 12.95 - 157 = 21.8$

Step 8. ANOVA table

Sources of variance	Sum of squares	Degrees of freedom	Mean square	Calculated Variance F	Table value of F at 1% level
Between Chemicals	12.95	$m-1$ 4 – 1 = 3	$\frac{12.95}{3}$ $= 4.31$	$\frac{4.31}{1.81} = 2.38$	$F_{0.01}(3,12)$ $= 5.95$

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Between Cloths of Bolts	157	$n-1$ 5 – 1 = 4	$\frac{157}{4}$ $=39.25$	$\frac{39.25}{1.81}=21.68$	$F_{0.01}(4,12)$ $=5.41$
Error	21.8	$(m-1)(n-1)$ = 12	$\frac{21.8}{12}$ $=1.81$		
Total	191.75	$N-1=19$			

Step 7 : Considering the difference between chemicals, we find that, calculated value of $F=2.38<$ tabulated value of $F_{1\%}=5.95$, we accept H_{01} : (the chemicals do not differ significantly)

Considering the difference between Bolts, we find that, calculated value of $F=21.68>$ tabulated value of $F_{1\%}=5.41$, we reject H_{02} : (the Cloths of Bolts differ significantly)

8. Three varieties A, B, C of a crop are tested in a randomized block design with 4 replications. The plot yields in pounds are as follows:

A 6	**C 5**	**A 8**	**B 9**
C 8	**A 4**	**B 6**	**C 9**
B 7	**B 6**	**C 10**	**A 6**

Analyze experimental yield and stat your conclusion. **(QC 53250 MA 6452 MAY 19)**

H_{01} (the varieties of crops do not differ significantly with respect to yield)

H_{02} : (the blocks do not differ significantly with respect to yield)

Rewriting the data such that the rows represent the blocks and the columns represent the varieties of crops, we have

Block	Variety of Crops		
	A	B	C
1	6	7	8
2	4	6	5
3	8	6	10
4	6	9	9

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Crops / Blocks	A	B	C	T_i	$\frac{T_i^2}{k}$	$\sum_i x_{ij}^2$
1	6	7	8	21	$\frac{(21)^2}{3}=147$	149
2	4	6	5	15	$\frac{(15)^2}{3}=75$	77
3	8	6	10	24	$\frac{(24)^2}{3}=192$	200
4	6	9	9	24	$\frac{24^2}{3}=192$	198
T_j	24	28	32	$T=84$	$\sum \frac{T_i^2}{k}=606$	$\sum x_{ij}^2=624$
$\frac{T_j^2}{h}$	$\frac{(24)^2}{4}=144$	$\frac{28^2}{4}=196$	$\frac{(32)^2}{4}=256$	$\sum \frac{T_j^2}{h}=596$		
$\sum_j x_{ij}^2$	152	202	270	$\sum x_{ij}^2=624$		

Correction Factor $\frac{\mathrm{T}^2}{\mathrm{N}}=\frac{(84)^2}{12}=588$

Total sum of squares $Q=\sum \sum x_{ij}^2-\frac{T^2}{N}=624-588=36$

Sum of squares between blocks $Q_1=\sum \frac{T_i^2}{k}-\frac{T^2}{N}=606-588=18$

Sum of squares between crops $Q_2=\sum \frac{T_j^2}{h}-\frac{T^2}{N}=596-588=8$

Error sum of square $Q_3=Q-Q_1-Q_2=36-18-8=10$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Blocks)	$Q_1=18$	$h-1=3$	$\frac{18}{3}=6$	$\frac{6}{1.67}=3.6$
Between Columns (Crops)	$Q_2=8$	$k-1=2$	$\frac{8}{2}=4$	$\frac{4}{1.67}=2.4$
Error	$Q_3=10$	$(h-1)(k-1)=6$	$\frac{10}{6}=1.67$	-
Total	$Q=36$	$hk-1=11$	-	-

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From F – table, $F_{5\%}(v_1 = 3, v_2 = 6) = 4.76$ and $F_{5\%}(v_1 = 2, v_2 = 6) = 5.14$

Considering the difference between rows, we find that, calculated value of $F = 3.6 <$ tabulated value of $F_{5\%} = 4.76$, we accept H_{01} : (the blocks do not differ significantly with respect to yield)

Considering the difference between columns, we find that, calculated value of $F = 2.4 <$ tabulated value of $F_{5\%} = 5.14$, we accept H_{02} : (the varieties of crops do not differ significantly with respect to yield)

9. **Three varieties of a crop are tested in a randomized block design with four replications, the layout being given as below. The yields are given in Kilograms. Analyze for significance.**

C 48	**A 51**	**B 52**	**A 49**
A 47	**B 49**	**C 52**	**C 51**
B 49	**C 53**	**A 49**	**B 50**

(QC 60045 MA3251 APR 22)

H_{01} (the blocks do not differ significantly with respect to yield)

H_{02} : (the varieties of crops do not differ significantly with respect to yield)

Rewriting the data such that the rows represent the blocks and the columns represent the varieties of the crop, we have

		Crops A	Crops B	Crops C
Blocks	1	47	49	48
	2	51	49	53
	3	49	52	52
	4	49	50	51

Shifting the origin to 50 and find out the new values of x_{ij}

Crops / Blocks	A	B	C	T_i	$\frac{T_i^2}{k}$	$\sum_i x_{ij}^2$
1	−3	−1	−2	−6	$\frac{(-6)^2}{3} = 12$	14
2	1	−1	3	3	$\frac{(3)^2}{3} = 3$	11
3	−1	2	2	3	$\frac{(3)^2}{3} = 3$	9

4	-1	0	1	0	$\frac{0^2}{3}=0$	2
T_j	-4	0	4	$T=0$	$\sum \frac{T_i^2}{k}=18$	$\sum x_{ij}^2=36$
$\frac{T_j^2}{h}$	$\frac{(-4)^2}{4}=4$	$\frac{0^2}{4}=0$	$\frac{(4)^2}{4}=4$	$\sum \frac{T_j^2}{h}=8$		
$\sum_j x_{ij}^2$	12	6	18	$\sum x_{ij}^2=36$		

Total sum of squares $Q=\sum\sum x_{ij}^2-\frac{T^2}{N}=36-\frac{0^2}{12}=36$

Sum of squares between blocks $Q_1=\sum \frac{T_i^2}{k}-\frac{T^2}{N}=18-\frac{0^2}{12}=18$

Sum of squares between crops $Q_2=\sum \frac{T_j^2}{h}-\frac{T^2}{N}=8-\frac{0^2}{12}=8$

Error sum of square $Q_3=Q-Q_1-Q_2=36-18-8=10$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Blocks)	$Q_1=18$	$h-1=3$	$\frac{18}{3}=6$	$\frac{6}{1.67}=3.6$
Between Columns (Crops)	$Q_2=8$	$k-1=2$	$\frac{8}{2}=4$	$\frac{4}{1.67}=2.4$
Error	$Q_3=10$	$(h-1)(k-1)=6$	$\frac{10}{6}=1.67$	-
Total	$Q=36$	$hk-1=11$	-	-

From $F-$table, $F_{5\%}\left(v_1=3,v_2=6\right)=4.76$ and $F_{5\%}\left(v_1=2,v_2=6\right)=5.14$
Considering the difference between rows, we find that, calculated value of $F=3.6<$ tabulated value of $F_{5\%}=4.76$, we accept H_{01}: (the blocks do not differ significantly with respect to yield)

Considering the difference between columns, we find that, calculated value of $F=2.4<$ tabulated value of $F_{5\%}=5.14$, we accept H_{02}: (the varieties of crops do not differ significantly with respect to yield)

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10. The result of an RBD experiment on 3 blocks with 4 treatments A, B, C, D are tabulated here. Carry out an analysis of variance.

		Treatment Effects			
Blocks	**I**	**A36**	**D35**	**C21**	**B36**
	II	**D32**	**B29**	**A28**	**C31**
	III	**B28**	**C29**	**D29**	**A26**

(QC 57506 MA 6452 MAY 16)

H_{01} (the blocks do not differ significantly with respect to treatment effects)

H_{02} :(the treatments do not differ significantly with respect to treatment effects)

Rewriting the data such that the rows represent the blocks and the columns represent the varieties of crops.

Block	Treatments A	B	C	D
I	36	36	21	35
II	28	29	31	32
III	26	28	29	29

Calculations are done by taking 30 as origin, we have

Treatments Blocks	A	B	C	D	T_i	$\frac{T_i^2}{k}$	$\sum_i x_{ij}^2$
I	6	6	–9	5	8	$\frac{(8)^2}{4}=16$	178
II	–2	–1	1	2	0	$\frac{(0)^2}{4}=0$	10
III	–4	–2	–1	–1	–8	$\frac{(-8)^2}{4}=16$	22
T_j	0	3	–9	6	$T=0$	$\sum\frac{T_i^2}{k}=32$	$\sum x_{ij}^2=210$
$\frac{T_j^2}{h}$	$\frac{(0)^2}{3}=0$	$\frac{3^2}{3}=3$	$\frac{(-9)^2}{3}=27$	$\frac{6^2}{3}=12$	$\sum\frac{T_j^2}{h}=42$		
$\sum_j x_{ij}^2$	56	41	83	30	$\sum x_{ij}^2=210$		

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Correction Factor $\frac{T^2}{N} = \frac{(0)^2}{12} = 0$

Total sum of squares $Q = \sum \sum x_{ij}^2 - \frac{T^2}{N} = 210 - 0 = 210$

Sum of squares between blocks $Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 32 - 0 = 32$

Sum of squares between treatments $Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 42 - 0 = 42$

Error sum of square $Q_3 = Q - Q_1 - Q_2 = 210 - 32 - 42 = 136$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Blocks)	$Q_1 = 32$	$h-1=2$	$\frac{32}{2}=16$	$\frac{22.6}{16}=1.41$
Between Columns (Treatments)	$Q_2 = 42$	$k-1=3$	$\frac{42}{3}=14$	$\frac{22.6}{14}=1.61$
Error	$Q_3 = 136$	$(h-1)(k-1)=6$	$\frac{136}{6}=22.6$	-
Total	$Q = 210$	$hk-1=11$	-	-

From $F-$table, $F_{5\%}(v_1 = 6, v_2 = 2) = 19.33$ and $F_{5\%}(v_1 = 6, v_2 = 3) = 8.94$
Considering the difference between rows, we find that, calculated value of $F = 1.41 <$ tabulated value of $F_{5\%} = 19.33$, we accept H_{01} : (the blocks do not differ significantly with respect to treatment effects)

Considering the difference between columns, we find that, calculated value of $F = 1.61 <$ tabulated value of $F_{5\%} = 8.94$, we accept H_{02} : (the varieties of treatments do not differ significantly with respect to treatment effects)

11. Four air conditioning compressor designs were tested in four different regions of India. The test was repeated by installing additional air conditioners in a second cooling season. The following are the times to failure (to the nearest month) of each compressor tested.

	Replicate 1 Designs				Replicate 2 Designs			
	A	**B**	**C**	**D**	**A**	**B**	**C**	**D**
Northeast	**58**	**35**	**72**	**61**	**49**	**24**	**60**	**64**

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Region	**Southeast**	**40**	**18**	**54**	**38**	**38**	**22**	**64**	**50**
	Northwest	**63**	**44**	**81**	**52**	**59**	**16**	**60**	**48**
	Southwest	**36**	**9**	**47**	**30**	**29**	**13**	**52**	**41**

Test at the 0.05 level of significance whether the difference among the means determined for designs, for regions, and for replicates are significant and for significance of the interaction between compressor designs and regions. **(QC 50782 MA 6452 NOV 17)**

H_{01} : There is no significant difference between Designs

H_{02} : There is no significant difference between Regions

H_{03} : There is no significant difference between Replicates

H_{04} : There is no significant difference within Regions and Designs

Find the sum of values of for the given data:

Replicate 1

	C1	C2	C3	C4	T_i	$\frac{T_i^2}{n}$
R1	58	35	72	61	226	12769
R2	40	18	54	38	150	5625
R3	63	44	81	52	240	14400
R4	36	9	47	30	122	3721
T_j	197	106	254	181	**738**	**36515**
$\frac{T_j^2}{n}$	9702	2809	16129	8190	**36830**	

Replicate 2

	C1	C2	C3	C4	T_i	$\frac{T_i^2}{n}$
R1	49	24	60	64	197	9702
R2	38	22	64	50	174	7569
R3	59	16	60	48	183	8372
R4	29	13	52	41	135	4556
T_j	175	75	236	203	**689**	**30199**
$\frac{T_j^2}{n}$	7656	1406	13924	10302	**33288**	

Construct the following two way analysis table

	A	B	C	D	Total
North East	107	59	132	125	423
South East	78	40	118	88	324
North West	122	60	141	100	423
South West	65	22	99	71	257
Total	372	181	490	384	**1427**

Correction factor $\frac{T^2}{N} = \frac{1427^2}{32} = 63635$

https://doi.org/10.5281/zenodo.15289873

Total sum of squares $Q = \sum \sum \sum x_{ij}^2 - \frac{T^2}{N} = 73667 - 63635 = 10032$

Sum of squares of regions(R) $Q_1 = \sum \frac{T_i^2}{n} - \frac{T^2}{N} = \left[\frac{423^2}{8} + \frac{324^2}{8} + \frac{423^2}{8} + \frac{257^2}{8} \right] - 63635 = 2475$

Sum of squares of designs(C) $Q_2 = \sum \frac{T_j^2}{n} - \frac{T^2}{N} = \left[\frac{372^2}{8} + \frac{181^2}{8} + \frac{490^2}{8} + \frac{384^2}{8} \right] - 63635 = 6203$

Sum of square between replicates $Q_3 = \sum \frac{T_k^2}{n} - \frac{T^2}{N} = \left[\frac{738^2}{16} + \frac{689^2}{16} \right] - 63635 = 75$

Sum of square between designs and regions

$$Q_4 = \sum \frac{T_{ij}^2}{n} - Q_1 - Q_2 - \frac{T^2}{N} = \begin{bmatrix} \frac{107^2}{2} + \frac{59^2}{2} + \frac{132^2}{2} + \frac{125^2}{2} + \frac{78^2}{2} \\ + \frac{40^2}{2} + \frac{118^2}{2} + \frac{88^2}{2} + \frac{122^2}{2} + \frac{60^2}{2} \\ + \frac{141^2}{2} + \frac{100^2}{2} + \frac{65^2}{2} + \frac{22^2}{2} + \frac{99^2}{2} + \frac{71^2}{2} \end{bmatrix} - 2475 - 6203 - 63635 = 311$$

Error sum of squares $Q_5 = Q - Q_1 - Q_2 - Q_3 - Q_4 = 10032 - 2475 - 6203 - 75 - 311 = 968$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Regions)	$Q_1 = 2475$	$r-1=3$	$\frac{2475}{3} = 825$	$\frac{825}{65} = 12.6$
Between Columns (Designs)	$Q_2 = 6203$	$c-1=3$	$\frac{6203}{3} = 2068$	$\frac{2068}{65} = 31.8$
Between Replicates	$Q_3 = 75$	$n-1=1$	$\frac{75}{1} = 75$	$\frac{75}{65} = 1.15$
Between Designs and Regions	$Q_4 = 311$	$(r-1)(c-1) = 9$	$\frac{311}{9} = 35$	$\frac{65}{35} = 1.85$
Error	$Q_5 = 968$	$(r-1)(rc-1) = 15$	$\frac{968}{15} = 65$	-
Total	$Q = 10032$	$2N-1 = 31$	-	-

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Considering the difference between Regions, we find that, calculated value of $F = 12.6 >$ tabulated value of $F_{5\%}(3,15) = 3.29$, we reject H_{01}: (the regions differ significantly)

Considering the difference between Designs, we find that, calculated value of $F = 31.8 >$ tabulated value of $F_{5\%}(3,15) = 3.29$, we reject H_{02} : (the design differ significantly)

Considering the difference between replicates, we find that, calculated value of $F = 1.15 <$ tabulated value of $F_{5\%}(1,15) = 4.54$, we accept H_{03} :(the replicates does not differ significantly)

Considering the difference between designs and regions, we find that, calculated value of $F = 1.85 <$ tabulated value of $F_{5\%}(15,9) = 3.01$, we accept H_{04} :(the design and region does not differ significantly)

Considering the main factors (regions and designs), the above problem can be solved in the following way also:

We will take only the total of the replication values corresponding to each combination of factors. Rewrite the table with the total of replication values in the two factor table.

Construct the following two way analysis table

Regions	Replicate 1	Replicate 2
North East	226	197
South East	150	174
North West	240	183
South West	122	135

Here $h = 4, \quad k = 2, \quad N = 8$

H_{01} : There is no significant difference between Designs

H_{02} : There is no significant difference between Regions

Calculation Table:

Replication Regions	1	2	T_i	$\frac{T_i^2}{k}$	$\sum_i x_{ij}^2$
North East	226	197	423	89465	89885
South East	150	174	324	52488	52776
North West	240	183	423	89465	91089
South West	122	135	257	33025	33109

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T_j	738	689	$T = 1427$	$\sum \frac{T_i^2}{k} =$ 264443	$\sum\sum x_{ij}^2 =$ 266859
$\frac{T_j^2}{h}$	136161	118680	$\sum \frac{T_j^2}{h} =$ 254841		

Correction factor $\frac{T^2}{N} = \frac{1427^2}{8} = 254541$

Total sum of squares $Q = \sum \sum x_{ij}^2 - \frac{T^2}{N} = 266859 - 254541 = 12318$

Sum of squares of regions(R) $Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 264443 - 254541 = 9902$

Sum of squares of designs(C) $Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 254841 - 254541 = 300$

Error sum of squares $Q_3 = Q - Q_1 - Q_2 = 12318 - 9902 - 300 = 2116$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Regions)	$Q_1 = 9902$	$h-1=3$	$\frac{9902}{3} = 3301$	$\frac{3301}{705} = 4.6$
Between Columns (Designs)	$Q_2 = 300$	$k-1=1$	$\frac{300}{1} = 300$	$\frac{705}{300} = 2.35$
Error	$Q_5 = 2116$	$(h-1)(k-1) = 3$	$\frac{2116}{3} = 705$	-
Total	$Q = 12318$	7	-	-

Considering the difference between Regions, we find that, calculated value of $F = 4.6 <$ tabulated value of $F_{5\%}(3,3) = 9.28$, we accept H_{01} : (the regions does not differ significantly)

Considering the difference between Designs, we find that, calculated value of $F = 2.35 <$ tabulated value of $F_{5\%}(3,1) = 216$, we accept H_{02} : (the design does not differ significantly)

Latin Square Design

Explain about Latin Square Design. MA 8452 APR 22

Consider an agricultural experiment, in which n^2 plots are taken and arranged in the form of an $n \times n$ square, such that the plots in each row will be

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homogeneous with respect to one factor, say, soil fertility and plots in each column will be homogeneous with respect to another factor, say, seed quality.

Then n treatments are given to these plots such that each treatment occurs only once in each row and column. The various possible arrangements obtained in this manner are known as Latin square of order n. This design of experiment is called Latin Square Design.

How to construct Latin Square? MA 6452 MAY 19

1. Compare Randomized Block Design over Latin Square Design. **(QC 60045 MA3251 APR 22)**

a. The number of replication of each treatment is equal to number of treatments in LSD, whereas there is no such restriction on treatments and replication in RBD.
b. LSD can be performed on a square field, while RBD can be performed either on a square or rectangular field
c. LSD is suitable for the case where the number of treatments is between 5 and 12, whereas RBD can be used for any number of treatments.

2. Is 2×2 Latin Square Design possible? Why?. **(QC 57506 MA 6452 MAY 16)**

A 2×2 Latin Square Design is not possible. Because the degree of freedom for error is $(2-1)(2-2)=0$ and hence comparisons are not possible.

Solved Problems

1. The following data resulted from an experiment to compare three burners B_1, B_2 and B_3. Use the Latin square design to test the hypothesis that there is no difference between the burners.

	Engine-1	Engine-2	Engine-3
Day-1	**B_1 – 16**	**B_2 - 17**	**B_3 - 20**
Day-2	**B_2 – 16**	**B_3 - 21**	**B_1 - 15**
Day-3	**B_3 – 15**	**B_1 - 12**	**B_2 - 13**

(QC 60045 MA3251 APR 22)

H_{01} : (the days do not differ significantly with respect to the burners)

H_{02} : (the engines do not differ significantly with respect to the burners)

H_{03} : (the burners do not differ significantly among themselves)

Shifting the origin to 16 and find out the new values of x_{ij}

Blocks	E1	E2	E3	T_i	$\frac{T_i^2}{k}$	$\sum_i x_{ij}^2$

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D1	0	1	4	5	$\frac{(5)^2}{3}=8.33$	17
D2	0	5	-1	4	$\frac{(4)^2}{3}=5.33$	26
D3	-1	-4	-3	-8	$\frac{(-8)^2}{3}=21.33$	26
T_j	-1	2	0	$T=1$	$\sum\frac{T_i^2}{k}=35$	
$\frac{T_j^2}{h}$	$\frac{(-1)^2}{3}=0.33$	$\frac{2^2}{3}=1.33$	$\frac{(0)^2}{3}=0$	$\sum\frac{T_j^2}{h}=1.66$		
$\sum_j x_{ij}^2$	1	42	26	$\sum x_{ij}^2=69$		

Re-arrange the data according to the burners, we have

Burner	x_k			T_k	$\frac{T_k^2}{n}$
B_1	0	-1	-4	-5	$\frac{(-5)^2}{3}=8.33$
B_2	1	0	-3	-2	$\frac{(-2)^2}{3}=1.33$
B_3	4	5	-1	8	$\frac{(8)^2}{3}=21.33$
	Total			$T=1$	$\sum\frac{T_k^2}{n}=31$

$$Q=\sum\sum x_{ij}^2-\frac{T^2}{N}=69-\frac{1}{9}=68.89 \qquad Q_1=\sum\frac{T_i^2}{n}-\frac{T^2}{N}=35-\frac{1}{9}=34.89$$

$$Q_2=\sum\frac{T_j^2}{n}-\frac{T^2}{N}=1.67-\frac{1}{9}=1.56 \qquad Q_3=\sum\frac{T_k^2}{n}-\frac{T^2}{N}=31-\frac{1}{9}=30.89$$

$$Q_4=Q-Q_1-Q_2-Q_3=68.89-34.89-1.56-30.89=1.55$$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Days)	$Q_1=34.89$	$n-1=2$	$\frac{34.89}{2}=17.445$	$\frac{17.445}{0.775}=22.51$
Between Columns (Engines)	$Q_2=1.56$	$n-1=2$	$\frac{1.,56}{2}=0.78$	$\frac{0.78}{0.775}=1.01$

Between Letters (Burners)	$Q_3 = 30.89$	$n-1=2$	$\frac{30.89}{2} = 15.445$	$\frac{15.445}{0.775} = 19.93$
Residual	$Q_4 = 1.55$	$(n-1)(n-2) = 2$	$\frac{1.55}{2} = 0.775$	-
Total	$Q = 68.89$	$n^2 - 1 = 8$	-	-

From F – table, $F_{5\%}\left(v_1 = 2, v_2 = 2\right) = 19$

Considering the difference between Days, we find that, calculated value of $F = 22.51 >$ tabulated value of $F_{5\%} = 19$, we reject H_{01}: (the days differ significantly)

Considering the difference between Engines, we find that, calculated value of $F = 1.01 <$ tabulated value of $F_{5\%} = 19$, we accept H_{02}: (the engines do not differ significantly)

Considering the difference between letters, we find that, calculated value of $F = 19.93 >$ tabulated value of $F_{5\%} = 19$, we reject H_{03}: (the burners differ significantly)

2. A Latin square design was used to compare the bond strengths of gold semi conductor lead wires bounded to the lead terminal by 5 different methods, A, B, C D and E. The bonds were made by 5 different operators and the devices were encapsulated using 5 different plastics. With the following results, expressed as pounds of force required to break the bond.

	Operator				
Plastics	**1**	**2**	**3**	**4**	**5**
1	**A 3**	**B 2.4**	**C 1.9**	**D 2.2**	**E 1.7**
2	**B 2.1**	**C 2.7**	**D 2.3**	**E 2.5**	**A 3.1**
3	**C 2.1**	**D 2.6**	**E 2.5**	**A 2.9**	**B 2.1**
4	**D 2**	**E 2.5**	**A 3.2**	**B 2.5**	**C 2.2**
5	**E 2.1**	**A 3.6**	**B 2.4**	**C 2.4**	**D 2.1**

Analyse these results and test with 0.01 level of significance. **(QC 41313 MA 6452 MAY 18)**

H_{01}: (the plastics do not differ significantly with respect to the methods of testing the bonds)

H_{02}: (the operators do not differ significantly with respect to the methods of testing the bonds.)

H_{03}: (the methods of testing the bonds do not differ significantly among themselves)

Tabulate the values of x_{ij} as follows:

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Operators Plastics	O1	O2	O3	O4	O5	T_i	$\frac{T_i^2}{n}$	$\sum_i x_{ij}^2$
P1	3	2.4	1.9	2.2	1.7	11.2	$\frac{(11.2)^2}{5} = 25.1$	26.1
P2	2.1	2.7	2.3	2.5	3.1	12.7	$\frac{(12.7)^2}{5} = 32.3$	32.85
P3	2.1	2.6	2.5	2.9	2.1	12.2	$\frac{(12.2)^2}{5} = 29.7$	30.24
P4	2	2.5	3.2	2.5	2.2	12.4	$\frac{(12.4)^2}{5} = 30.8$	31.58
P5	2.1	3.6	2.4	2.4	2.1	12.6	$\frac{(12.6)^2}{5} = 31.8$	33.3
T_j	11.3	13.8	12.3	12.5	11.2	$T = 61.1$	$\sum \frac{T_i^2}{n} = 150$	$\sum x_{ij}^2 = 154$
$\frac{T_j^2}{n}$	$\frac{(11.3)^2}{5} = 25.5$	$\frac{13.8^2}{5} = 38.1$	$\frac{12.3^2}{5} = 30.3$	$\frac{12.5^2}{5} = 31.3$	$\frac{11.2^2}{5} = 25.1$	$\sum \frac{T_j^2}{n} = 150$		
$\sum_j x_{ij}^2$	26.2	39	31.2	31.5	26	$\sum x_{ij}^2 = 154$		

Re-arrange the data according to the methods of testing, we have

Method of testing	x_k					T_k	$\frac{T_k^2}{n}$
A	3	3.1	2.9	3.2	3.6	15.8	$\frac{(15.8)^2}{5} = 49.9$

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B	2.4	2.1	2.1	2.5	2.4	11.5	$\frac{(11.5)^2}{5}=26.5$
C	1.9	2.7	2.1	2.2	2.4	11.3	$\frac{(11.3)^2}{5}=25.5$
D	2.2	2.3	2.6	2	2.1	11.2	$\frac{(11.2)^2}{5}=25.1$
E	1.7	2.5	2.5	2.5	2.1	11.3	$\frac{(11.3)^2}{5}=25.5$
	Total					$T=61.1$	$\sum\frac{T_k^2}{n}=152.5$

Total sum of squares $Q=\sum\sum x_{ij}^2-\frac{T^2}{N}=154-\frac{61.1^2}{25}=4.67$

Sum of squares of plastics(rows) $Q_1=\sum\frac{T_i^2}{n}-\frac{T^2}{N}=150-\frac{61.1^2}{25}=0.6716$

Sum of squares of operators(columns) $Q_2=\sum\frac{T_j^2}{n}-\frac{T^2}{N}=150-\frac{61.1^2}{25}=0.6716$

Sum of square within letters $Q_3=\sum\frac{T_k^2}{n}-\frac{T^2}{N}=152.5-\frac{61.1^2}{25}=3.17$

Error sum of squares $Q_4=Q-Q_1-Q_2-Q_3=4.67-0.6716-0.6716-3.17=0.1568$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows (Plastics)	$Q_1=0.6716$	$n-1=4$	$\frac{0.6716}{4}=0.167$	$\frac{0.167}{0.013}=12.8$

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Between Columns (Operators)	$Q_2 = 0.6716$	$n-1=4$	$\frac{0.6716}{4} = 0.167$	$\frac{0.167}{0.013} = 12.8$
Between Letters (Method of testing)	$Q_3 = 3.17$	$n-1=4$	$\frac{3.17}{4} = 0.793$	$\frac{0.793}{0.013} = 61$
Residual	$Q_4 = 0.1568$	$(n-1)(n-2) = 12$	$\frac{0.1568}{12} = 0.013$	-
Total	$Q = 4.67$	$n^2 - 1 = 24$	-	-

From F – table, $F_{1\%}(v_1 = 4, v_2 = 12) = 5.41$

Considering the difference between Plastics, we find that, calculated value of $F = 12.8 >$ tabulated value of $F_{1\%} = 5.41$, we reject H_{01} : (the plastics differ significantly)

Considering the difference between Operators, we find that, calculated value of $F = 12.8 >$ tabulated value of $F_{1\%} = 5.41$, we reject H_{02} : (the operators differ significantly)

Considering the difference between letters, we find that, calculated value of $F = 61 >$ tabulated value of $F_{1\%} = 5.41$, we reject H_{03} : (the methods differ significantly)

3. The following is the Latin square layout of a design when 4 varieties of seeds are tested. Set up the ANOVA table and state your conclusions. **(QC 20753 MA 6452 NOV 18)**

A 18	**C 21**	**D 25**	**B 11**
D 22	**B 12**	**A 15**	**C 19**
B 15	**A 20**	**C 23**	**D 24**
C 22	**D 21**	**B 10**	**A 17**

H_{01} : (the rows do not differ significantly with respect to the growth of the seeds)

H_{02} : (the columns do not differ significantly with respect to the growth of the seeds)

H_{03} : (the varieties of seed do not differ significantly among themselves)

Tabulate the values of x_{ij} by taking 17 as origin as follows:

	C1	C2	C3	C4	T_i	$\frac{T_i^2}{n}$	$\sum_i x_{ij}^2$

R1	1	4	8	-6	7	$\frac{(7)^2}{4}=12.25$	117
R2	5	-5	-2	2	0	$\frac{(0)^2}{4}=0$	58
R3	-2	3	6	7	14	$\frac{(14)^2}{4}=49$	98
R4	5	4	-7	0	2	$\frac{(2)^2}{4}=1$	90
T_j	9	6	5	3	$T=23$	$\sum \frac{T_i^2}{n}=62.25$	$\sum x_{ij}^2=363$
$\frac{T_j^2}{n}$	$\frac{(9)^2}{4}$ $=20.25$	$\frac{6^2}{4}$ $=9$	$\frac{5^2}{4}$ $=6.25$	$\frac{3^2}{4}$ $=2.25$	$\sum \frac{T_j^2}{n}=37.75$		
$\sum_j x_{ij}^2$	55	66	153	89	$\sum x_{ij}^2=363$		

Re-arrange the data according to the varieties of seeds(letters) by taking 17 as origin, we have

Varieties of seeds	x_k				T_k	$\frac{T_k^2}{n}$
A	1	-2	3	0	2	$\frac{(2)^2}{4}=1$
B	-6	-5	-2	-7	-20	$\frac{(-20)^2}{4}=100$
C	4	2	6	5	17	$\frac{(17)^2}{4}=72.25$
D	8	5	7	4	24	$\frac{(24)^2}{4}=144$
	Total				$T=23$	$\sum \frac{T_k^2}{n}=317.25$

$$Q=\sum \sum x_{ij}^2-\frac{T^2}{N}=363-\frac{23^2}{16}=329.9$$

$$Q_1=\sum \frac{T_i^2}{n}-\frac{T^2}{N}=62.25-\frac{23^2}{16}=29.18$$

$$Q_2=\sum \frac{T_j^2}{n}-\frac{T^2}{N}=37.75-\frac{23^2}{16}=4.687$$

$$Q_3=\sum \frac{T_k^2}{n}-\frac{T^2}{N}=317.25-\frac{23^2}{16}=284.18$$

$$Q_4=Q-Q_1-Q_2-Q_3=329.9-29.18-4.687-284.18=11.85$$

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ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows	$Q_1 = 29.18$	$n-1=3$	$\frac{29.18}{3}=9.72$	$\frac{9.72}{1.975}=4.92$
Between Columns	$Q_2 = 4.687$	$n-1=3$	$\frac{4.687}{3}=1.562$	$\frac{1.975}{1.562}=1.264$
Between Letters (variety of seeds)	$Q_3 = 284.18$	$n-1=3$	$\frac{284.18}{3}=94.7$	$\frac{94.7}{1.975}=47.9$
Residual	$Q_4 = 11.85$	$(n-1)(n-2)=6$	$\frac{11.85}{6}=1.975$	-
Total	$Q = 329.9$	$n^2-1=15$	-	-

From $F-$table, $F_{5\%}\left(v_1=3, v_2=6\right)=4.76$ and $F_{5\%}\left(v_1=6, v_2=3\right)=8.94$

Considering the difference between rows, we find that, calculated value of $F=4.92>$ tabulated value of $F_{5\%}=4.76$, we reject H_{01}: (the rows differ significantly)

Considering the difference between columns, we find that, calculated value of $F=1.264<$ tabulated value of $F_{5\%}=8.94$, we accept H_{02}: (the columns do not differ significantly)

Considering the difference between variety of seeds(letters), we find that, calculated value of $F=47.9>$ tabulated value of $F_{5\%}=4.76$, we reject H_{03}: (the variety of seeds differ significantly)

4. A variable trial was conducted on wheat with 4 varieties in a Latin Square design. The plan of the experiment is given below. Analyse the data and interpret the result.

C 25	**B 23**	**A 20**	**D 20**
A 19	**D 19**	**C 21**	**B 18**
B 19	**A 14**	**D 17**	**C 20**
D 17	**C 20**	**B 21**	**A 15**

(QC 72071 MA 6452 MAY 17)

H_{01}: (the rows do not differ significantly with respect to the growth of the wheat)

H_{02}: (the columns do not differ significantly with respect to the growth of the wheat)

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H_{03} : (the varieties of wheat do not differ significantly among themselves)

Tabulate the values of x_{ij} by taking 19 as origin as follows:

	C1	C2	C3	C4	T_i	$\frac{T_i^2}{n}$	$\sum_i x_{ij}^2$
R1	6	4	1	1	12	$\frac{(12)^2}{4}=36$	54
R2	0	0	2	−1	1	$\frac{(1)^2}{4}=0.25$	5
R3	0	−5	−2	1	−6	$\frac{(-6)^2}{4}=9$	30
R4	−2	1	2	−4	−3	$\frac{(-3)^2}{4}=2.25$	25
T_j	4	0	3	−3	$T=4$	$\sum\frac{T_i^2}{n}=47.5$	$\sum x_{ij}^2=114$
$\frac{T_j^2}{n}$	$\frac{(4)^2}{4}$ $=4$	$\frac{0^2}{4}$ $=0$	$\frac{3^2}{4}$ $=2.25$	$\frac{(-3)^2}{4}$ $=2.25$	$\sum\frac{T_j^2}{n}=8.5$		
$\sum_j x_{ij}^2$	40	42	13	19	$\sum x_{ij}^2=114$		

Re-arrange the data according to the varieties of seeds(letters) by taking 19 as origin, we have

Varieties of seeds	x_k				T_k	$\frac{T_k^2}{n}$
A	1	0	−5	−4	−8	$\frac{(-8)^2}{4}=16$
B	4	−1	0	2	5	$\frac{(5)^2}{4}=6.25$
C	6	2	1	1	10	$\frac{(10)^2}{4}=25$
D	1	0	−2	−2	−3	$\frac{(-3)^2}{4}=2.25$
	Total				$T=4$	$\sum\frac{T_k^2}{n}=49.5$

$Q=\sum\sum x_{ij}^2-\frac{T^2}{N}=114-\frac{4^2}{16}=113$ $\qquad Q_1=\sum\frac{T_i^2}{n}-\frac{T^2}{N}=47.5-\frac{4^2}{16}=46.5$

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$$Q_2 = \sum \frac{T_j^2}{n} - \frac{T^2}{N} = 8.5 - \frac{4^2}{16} = 7.5 \qquad Q_3 = \sum \frac{T_k^2}{n} - \frac{T^2}{N} = 49.5 - \frac{4^2}{16} = 48.5$$

$$Q_4 = Q - Q_1 - Q_2 - Q_3 = 113 - 46.5 - 7.5 - 48.5 = 10.5$$

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Calculated F Value
Between Rows	$Q_1 = 46.5$	$n-1=3$	$\frac{46.5}{3} = 15.5$	$\frac{15.5}{1.75} = 8.85$
Between Columns	$Q_2 = 7.5$	$n-1=3$	$\frac{7.5}{3} = 2.5$	$\frac{2.5}{1.75} = 1.42$
Between Letters (variety of seeds)	$Q_3 = 48.5$	$n-1=3$	$\frac{48.5}{3} = 16.16$	$\frac{16.6}{1.75} = 9.48$
Residual	$Q_4 = 10.5$	$(n-1)(n-2) = 6$	$\frac{10.5}{6} = 1.75$	-
Total	$Q = 113$	$n^2 - 1 = 15$	-	-

From $F-$table, $F_{5\%}\left(v_1 = 3, v_2 = 6\right) = 4.76$

Considering the difference between rows, we find that, calculated value of $F = 8.85 >$ tabulated value of $F_{5\%} = 4.76$, we reject H_{01} : (the rows differ significantly)

Considering the difference between columns, we find that, calculated value of $F = 1.42 <$ tabulated value of $F_{5\%} = 4.76$, we accept H_{02} : (the columns do not differ significantly)

Considering the difference between variety of wheat(letters), we find that, calculated value of $F = 9.48 >$ tabulated value of $F_{5\%} = 4.76$, we reject H_{03} : (the variety of seeds differ significantly)

Factorial Designs

In factorial experiment, the effect of several variations are studied and the treatments being all the combinations of different factor under study. Here an attempt is made to estimate the effect of each factors and also the interaction effects, i.e. the variation in the effect of one factor as a result of different levels of other factors. If 2 factors are considered simultaneously, the design is called 2-factor factorial design.

What is a 2^2- factorial design. MA 6452 NOV 18

2^2 Factorial Design

Explain 2^2 factorial design with an

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Here we have two factors each at two levels (0,1) say, so that there are $2\times2=4$ treatment combinations in total. In Yates notation, let A and B indicate the names of two factors under study and let a and b denote one of the two levels of each of the corresponding factors and this will be called the second level. The four treatment combinations are as follows:

example.
MA 8452 APR 22

a_0b_0 (or) 1 – Factors A and B, both at first level

a_1b_0 (or) a – A at second level and B at first level

a_0b_1 (or) b – A at first level and B at second level

a_1b_1 (or) ab – A and B both at second level

These combinations can be compared by laying out the experiment in

(i) RBD with r replications, each replicate containing 4 units (or)

(ii) 4×4 LSD and ANOVA can be carried out accordingly.

Analysis of 2^2 Design

Factorial experiments are conducted in RBD in the usual manner except that in this case, the treatment sum of squares with 3 d.f into three orthogonal components each with 1 d.f. Factorial effect totals are given by the following table:

Yate's Method for 2^2 experiment

Treatment combinations (1)	Total yield from all blocks (2)	(3)	(4)	Effect Totals (5))=(4)²/4r
1	[1]	[1]+[a]	[1]+[a]+[b]+[ab]	G
a	[a]	[b]+[ab]	[ab]–[b]+[a]-[1]	[A]
b	[b]	[a]–[1]	[ab]+[b] – [a] – [1]	[B]
ab	[ab]	[ab]–[b]	[ab]–[b] – [a] + [1]	[AB]

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ANOVA table for the 2^2 - experiment

Source of Variation	d.f	SS	MSS	Calculated F-Value	Tabulated $F_{0.05}$ -Value
Blocks	$r-1$	BSS S_R^2			At the respective degrees of freedom
Treatments	$r-1$	TSS			
Main effect A	1	$S_A^2 = [A]^2 / 4r$	$\frac{SS}{d.f}$	$\frac{MSS}{S.E}, (Nr > Dr)$	
Main effect B	1	$S_B^2 = [B]^2 / 4r$			
Interaction AB	1	$S_{AB}^2 = [AB]^2 / 4r$			
Error	$3(r-1)$	S_E^2			
Total	15				

Where r is the common replication number.

Conclusion

If the calculated value of F is < the tabulated value F at certain level of significance, then the null hypothesis of the presence of the factorial effect is accepted. Otherwise it is rejected.

Advantages of factorial experiment

- It increases the scope of the treatment by giving information about the main factors and their interactions
- The various levels of one factor constitute replications of other factors and increase the amount of information obtained in all factors

Solved Problems

1. Find out the main effects and interactions in The following 2^2 – factorial experiment and write down the ANOVA table.

		1 00	a 10	b 01	ab 11
Block	I	64	25	30	60
	II	75	14	50	33
	III	76	12	41	17
	IV	75	33	25	10

(QC 27331 MA 6452 NOV 15)

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Re-arrange the data by subtracting 50 from each values.

Treatment→ combinations↓	B I	B II	B III	B IV	Treatment Totals T_i	T_i^2
1	14	25	26	25	90	8100
a	–25	–36	–38	–17	–116	13456
b	–20	0	–9	–25	–54	2916
ab	10	–17	–33	–40	–80	6400
Block Total B_j	–21	–28	–54	–57	**G = –160**	**30872**
B_j^2	441	784	2916	3249	**7390**	

H_0: The given data is homogeneous with respect to blocks and treatments

H_1: The given data is not homogeneous with respect to blocks and treatments

Here $N=16$ *and* $G=-160$. Correction Factor $=\frac{G^2}{N}=\frac{(-160)^2}{16}=1600$

x_1^2	x_2^2	x_3^2	x_4^2
196	625	676	625
625	1296	1444	289
400	0	81	625
100	289	1089	1600
T: 1321	T: 2210	T: 3290	T: 3139
$\sum\sum x_{ij}^2=9960$			

Total sum of squares $=\sum\sum x_{ij}^2-\frac{G^2}{N}=9960-1600=8360$

Treatment sum of squares $=\frac{1}{4}\sum T_i^2-\frac{G^2}{N}=\frac{1}{4}(30872)-1600=6118$

Block sum of squares $=\frac{1}{4}\sum B_j^2-\frac{G^2}{N}=\frac{1}{4}(7390)-1600=248$

Error Sum of Squares = Total SS – Treatment SS – BSS = 8360 – 6118 – 248 = 1994

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Now compute the factorial effect Totals by Yates method

Treatment combinations (1)	Total yield from all blocks (2)	(3)	Factorial effects total (4)	SS (5)=(4)²/4r
1	90	–26	–160(G)	1600(CF)
a	–116	–134	–232(A)	3364 S_A^2
b	–54	–206	–108(B)	729 S_B^2
ab	–80	–26	180(AB)	2025 S_{AB}^2

Now form ANOVA table for the 2^2 – experiment

Choose 5% level of significance

Source of Variation	d.f	SS	MSS	Calculated F-Value	Tabulated $F_{0.05}$-Value
Blocks	3	248	82.6	2.68	$F_{0.05}(9,3) = 8.8$
Treatments	3	6118	2039.3	9.2	$F_{0.05}(3,9) = 3.8$
A	1	3364	3364	15.1	$F_{0.05}(1,9) = 5.1$
B	1	729	729	3.29	$F_{0.05}(1,9) = 5.1$
AB	1	2020	2025	9.14	$F_{0.05}(1,9) = 5.1$
Error	9	1994	221.5		
Total	18				

Since calculated value of $F-$ (for blocks and main effect B) is less than the corresponding tabulated value, we may conclude that the blocks do not differ significantly.

But in other cases (treatments, main effect A and interaction effects) the computed value of $F-$is greater than the tabulated value, we conclude that these effects differ significantly.

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2 **An experiment was planned to study the effect of sulphate of potash and super phosphate on the yield of potatoes. All the combinations of 2 levels of super phosphate and 2 levels of sulphate of potash were studied in a randomized block design with 4 replications for each. The yields (per plot) obtained are given below.** **QC E3126 MA 2266 MAY 2010**

Block	**Yields**			
I	**(1)**	**k**	**p**	**kp**
	23	**25**	**22**	**38**
II	**p**	**(1)**	**k**	**kp**
	40	**26**	**36**	**38**
III	**(1)**	**k**	**kp**	**p**
	29	**20**	**30**	**20**
IV	**kp**	**k**	**p**	**(1)**
	34	**31**	**24**	**28**

Analyse the data and comment on your findings. $F_{0.05}(3,9)=3.86$, $F_{0.05}(1,9)=5.12$.

Re-arrange the data by subtracting 29 from each values.

Treatment→ combinations↓	B I	B II	B III	B IV	Treatment Totals T_i	T_i^2
1	–6	–3	0	–1	–10	100
k	–4	–7	–9	2	–4	16
p	–7	11	–9	–5	–10	100
kp	9	9	1	5	24	576
Block Total B_j	–8	24	–17	1	**G = 0**	**792**
B_j^2	64	576	289	1	**930**	

H_0: The given data is homogeneous with respect to blocks and treatments

H_1: The given data is not homogeneous with respect to blocks and treatments

Here $N=16$ *and* $G=0$. Correction Factor $=\dfrac{G^2}{N}=\dfrac{(0)^2}{16}=0$

x_1^2	x_2^2	x_3^2	x_4^2
36	9	0	1
16	49	81	4
49	121	81	25
81	81	1	25

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T: 182	T: 260	T: 163	T: 55
$\sum\sum x_{ij}^2 = 660$			

Total sum of squares $= \sum\sum x_{ij}^2 - \frac{G^2}{N} = 660 - 0 = 660$

Treatment sum of squares $= \frac{1}{4}\sum T_i^2 - \frac{G^2}{N} = \frac{1}{4}(792) - 0 = 198$

Block sum of squares $= \frac{1}{4}\sum B_j^2 - \frac{G^2}{N} = \frac{1}{4}(930) - 0 = 232.5$

Error Sum of Squares = Total SS – Treatment SS – BSS = 660 – 198 – 232.5 = 229.5

Computation of the factorial effect Totals by Yates method

Treatment combinations (1)	Total yield from all blocks (2)	(3)	Factorial effects total (4)	SS (5)=(4)²/4r
1	–10	–14	0(G)	0 (CF)
k	–4	14	40(A)	$100\, S_k^2$
p	–10	6	28(B)	$49\, S_p^2$
kp	24	34	28(AB)	$49\, S_{kp}^2$

Now form ANOVA table for the $2^2 -$ experiment
Choose 5% level of significance

Source of Variation	d.f	SS	MSS	Calculated F-Value	Tabulated $F_{0.05}$ -Value
Blocks	3	232.5	77.,5	3.03	$F_{0.05}(3,9) = 3.86$
Treatments	3	198	66	2.58	$F_{0.05}(3,9) = 3.86$
k	1	100	100	3.92	$F_{0.05}(1,9) = 5.12$
p	1	49	49	1.92	$F_{0.05}(1,9) = 5.12$
kp	1	49	49	1.92	$F_{0.05}(1,9) = 5.12$
Error	9	229.5	25.5		
Total	18	660			

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Since calculated value of $F-$ (in all cases) is less than the corresponding tabulated value, we may conclude that there is no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly.

EXERCISE

1 What do you understand by Design of Experiment? QC 21528 MA 2266 MAY 2013

2 Write down the ANOVA table for one way classification. QC 21528 MA 2266 MAY 2013

3 State any two advantages of a completely randomized design. QC 51579 MA 2266 MAY 2014

4 Write any two differences between RBD and CRD. QC 11395 MA 2266 MAY 2011

5 Discuss the advantages and disadvantages of RBD. QC E3126 MA 2266 MAY 2010

6 State the advantages of a factorial experiment over a simple experiment. QC E3126 MA 2266 MAY 2010

One Way Analysis

1 The following are the number of mistakes made in 5 successive days by 4 technicians working for a photographic laboratory test at a level of significance $\alpha = 0.01$. Test whether the difference among the four sample means can be attributed to chance. QC 11395 MA 2266 MAY 2011

Technician

I	II	III	IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

2 A completely randomized design experiment with 10 plots and 3 treatments gave the results given in the table below: Analyse the results for the effects of treatments.

Treatment	Replications			
A	5	7	1	3
B	4	4	7	
C	3	1	5	

Two Way Analysis

1 The following date represent the number of units of production per day turned out by 5 different workers using 4 different types of machines. QC 21528 MA 2266 MAY 2013

	Machine Type			
	A	B	C	D
1	44	38	47	36

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	2	46	40	52	43
Workers	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

2 Analyse the following RBD and find your conclusion QC 31528 MA 2266 NOV 2013

		Treatments			
		T1	T2	T3	T4
	B1	12	14	20	22
	B2	17	27	19	15
Blocks	B3	15	14	17	12
	B4	18	16	22	12
	B5	19	15	20	14

3 Four varieties A, B, C and D of a fertilizer are tested in a RBD with four replications. The plot yields in pounds are as follows: QC 51579 MA 2266 MAY 2014

A 12	D 20	C 16	B 10
D 18	A 14	B 11	C 14
B 12	C 15	D 19	A 13
C 16	B 11	A 15	D 20

Analyse the experimental yield.

Latin Square Design

1 The following is a Latin Square of a design when 4 varieties of seeds are being tested. Setup the analysis of variance table and state your conclusion. You may carry out suitable change of origin and scale. QC 21528 MA 2266 MAY 2013

A 105	B 95	C 125	D 115
C 115	D 125	A 105	B 105
D 115	C 95	B 105	A 115
B 95	A 135	D 95	C 115

2 Analyse the following of Latin Square experiment QC 21528 MA 2266 MAY 2013

		Column			
		1	2	3	4
	1	A 12	D 20	C 16	B 10
Row	2	D 18	A 14	B 11	C 14
	3	B 12	C 15	D 19	A 13
	4	C 16	B 11	A 15	D 20

3 The following is a Latin square of a design when 4 varieties of seed are being tested. Set up the analysis of variance table and state your conclusion. You can carry out the suitable change of origin and scale. QC 31528 MA 2266 NOV 2013

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A 110	B 100	C 130	D 120
C 120	D 130	A 110	B 110
D 120	C 100	B 110	A 120
B 100	A 140	D 100	C 120

4 Analyse the variance in the Latin square design of yields (in kgs) of paddy where P, Q, R, S denote the different methods of cultivation: QC 51579 MA 2266 MAY 2014

S 122	P 121	R 123	Q 122
Q 124	R 123	P 122	S 125
P 120	Q 119	S 120	R 121
R 122	S 123	Q 121	P 122

Examine whether the different method of cultivation have significantly different yields.

5. Perform Latin Square Experiment for the following:

Roman	I	II	III	Three equally spaced concentrations of poison as extracted From the scorpion fish
Arabic	1	2	3	Three equally spaced body weights for the animals tested
Latin	A	B	C	Three equally spaced time of storage of the poison before it is administered to the animals

	I	II	III
1	0.194 A	0.73 B	1.187 C
2	0.758 C	0.311 A	0.589 B
3	0.369 B	0.558 C	0.311 A

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UNIT III – SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

Introduction

The solution of equations of the form $f(x)=0$ occurs in many engineering applications. Equations are of two types in general. They are algebraic equations and transcendental equations. Equation involving functions like exponential, trigonometrical and logarithmic is called transcendental equation.

Algebraic Equations	Transcendental Equations
i. $2x^2+x-4=0$	i. $xe^x-2=0$
ii. $x^3-2x^2+3=0$	ii. $\cos x-2x+3=0$

The solution of the transcendental equations nor not integers, in general but real number. The approximated roots are found out by numerical method.

Theorem : If $f(x)$ is continuous in the interval (a,b) and if $f(a)$ & $f(b)$ are of opposite signs, then the equation $f(x)=0$ will have at least one root between a & b.

Note: In general, if $|f(a)|<|f(b)|$ the root is nearer to a.

Errors

While evaluating roots, round off and truncation errors may occurs.

The round off error is the quantity which is added to the computed value to make it the exact value.

Truncation errors are defined as those errors that result from using an approximation in the place of an exact mathematical procedure. Truncation error results from terminating after a finite number of terms.

Order of convergence of an iterative method: Let $x_0, x_1, \ldots, x_n$ be the successive approximations of the root α of $f(x)=0$. Let e_i be the error in the root x_i. Consider the successive errors $e_i=x_i-\alpha$ and $e_{i+1}=x_{i+1}-\alpha$. If $p\ge 1$ can be found out such that $|e_{i+1}|\le k.|e_i|^p$ where k is a positive constant for all , then p is called the order of convergence.

Note

1. If $p=2$, it is said to be quadratic convergence.
2. Quadratic convergence means that each error is proportional to the square of the preceding error. i.e. the number of significant figures doubles in each iteration.

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Bisection Method

(To find the root of algebraic or transcendental equation)
Working Rule: • Let $f(x)=0$ be the given equation • Choose two subsequent integers a & b such that $f(a)$ & $f(b)$ are of opposite signs. i.e. put $x=0, 1, 2, \ldots$, check the signs of $f(0),\ f(1),\ f(2), \ldots$ and find a & b. • Assume that $f(a)=-ve$ & $f(b)=+ve$. Then the root lies between a & b. • Then the first approximate root is given by $x_1=\frac{a+b}{2}$. • Find the sign of $f(x_1)$.. If $f(x_1)=-ve$ then, root lies between x_1 & b. Therefore second approximate root is given by $x_2=\frac{x_1+b}{2}$. • If $f(x_1)=+ve$ then root lies between a & x_1. Therefore second approximate root is given by $x_2=\frac{a+x_1}{2}$. • Continue this process till the required accuracy is reached

1. Find a root of the equation $x^3-4x-9=0$, using the bisection method correct to two decimal places. **(QC 20817 MA 8452 APR 2022)**

Let $f(x)=x^3-4x-9$

$f(2)=2^3-4(2)-9=-9$

$f(3)=3^3-4(3)-9=6$

Therefore, the root lies between $a=2$ and $b=3$.

Therefore by Bisection method

$$x_1=\frac{a+b}{2}=\frac{2+3}{2}=2.5$$

Now $f(2.5)=2.5^3-4(2.5)-9=-3.375$

Therefore, the root lies between $a=2.5$ and $b=3$

.

2. Find the real root of $\cos x-2x+3=0$ correct to 3 decimal places using iteration method.

Let $f(x)=\cos x-2x+3$

$f(0)\ =\ \cos 0-2(0)+3=\ 4$

$f(1)\ =\ \cos 1-2(1)+3=\ 1.5$

$f(2)\ =\ \cos 2-2(2)+3=\ -1.4$

Therefore, the root lies between $a=1$ and $b=2$.

Therefore by Bisection method

$$x_1=\frac{a+b}{2}=\frac{1+2}{2}=1.5$$

Now $f(1.5)=\cos 1.5-2(1.5)+3=0.0707$

https://doi.org/10.5281/zenodo.15289885

Therefore by Bisection method

$$x_2 = \frac{a+b}{2} = \frac{2.5+3}{2} = 2.75$$

Now $f(2.75) = 2.75^3 - 4(2.75) - 9 = 0.7968$

Therefore, the root lies between

$$a = 2.5 \ and \ b = 2.75.$$

Therefore by Bisection method

$$x_3 = \frac{a+b}{2} = \frac{2.5+2.75}{2} = 2.625$$

Therefore the approximate root is $x = 2.625$

Therefore, the root lies between $a = 1.5 \ and \ b = 2$

.

Therefore by Bisection method

$$x_2 = \frac{a+b}{2} = \frac{1.5+2}{2} = 1.75$$

Now $f(1.75) = \cos 1.75 - 2(1.75) + 3 = -0.678$

Therefore, the root lies between

$$a = 1.5 \ and \ b = 1.75.$$

Therefore by Bisection method

$$x_3 = \frac{a+b}{2} = \frac{1.5+1.75}{2} = 1.625$$

Therefore the approximate root is $x = 1.625$

.

Fixed Point Iteration Method or Iterative Method

(To find the root of algebraic or transcendental equation)

Working Rule:

- Let $f(x) = 0$ be the given equation
- Choose two subsequent integers a & b such that $f(a)$ & $f(b)$ are of opposite signs. i.e. put $x = 0, 1, 2, \ldots.$, check the signs of $f(0), f(1), f(2), \ldots.$ and find a & b.
- Assume that $f(a) = -ve$ & $f(b) = +ve$. Then the root lies between a & b.
- Then choose initial root $x_0 \in (a, b)$. Hint: If $|f(a)| < |f(b)|$, then the root is nearer to $'a'$.
- Rewrite $f(x)$ as $x = \phi(x)$ such that $|\phi'(x_0)| < 1$.
- Then the successive roots are given by $x_{n+1} = \phi(x_n)$
- Continue this process till the required accuracy is reached

Results:

- The sufficient condition for the convergence is $|\phi'(x_0)| < 1$ where $x_0 \in (a, b)$
- If $|\phi'(x_0)|$ is very small, the convergence will be rapid
- Order of convergence of this method is 1
- This method is useful if the equation is given in the form of infinite series.

  Example: $x - 1 - \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^3} - \ldots\ldots = 0$

1. **Derive order of fixed point iteration method.** **QC 20817 MA 8452 APR 2022**

Let α be the root of the equation $x=\phi(x)$. Therefore $\alpha=\phi(\alpha)$ ....(1)

Let x_{n-1} and x_n be the approximations of α. Therefore $x_n=\phi(x_{n-1})$ .....(2).

From (1) and (2), we have $x_n-\alpha=\phi(x_{n-1})-\phi(\alpha)$ ....(3)

By Mean Value Theorem, we have $\phi(x_{n-1})-\phi(\alpha)=\left(x_{n-1}-\alpha\right)\phi'(\theta),\ x_{n-1}<\theta<\alpha.$ .....(4)

Substitute (4) in (3), we get $x_n-\alpha=\left(x_{n-1}-\alpha\right)\phi'(\theta).$ .....(5)

Let k be the maximum value of $|\phi'(x)|$ in the interval I. i.e. $|\phi'(x)|\le k$ for all x in I .

Then from (5), we get $|x_n-\alpha|\le k|x_{n-1}-\alpha|$ i.e. $|e_n|\le k|e_{n-1}|$.

Here, $p=1$ (power of $|e_{n-1}|$) the order of convergence of the iterative method is 1.

2. **State the order of convergence of iterative method.** **QC 53252 MA 6459 MAY 2019**

The order of convergence of the iterative method is 1.

3. **Write the condition for convergence of iteration method.** **QC 90346 MA 8491 DEC 2019**

Let $f(x)=0$ be the given equation and let it be written as $x=\phi(x)$. If $|\phi'(x_0)|<1$, then the method will converge.

.

4. **Find a positive root of $f(x)=2x-\log_{10}x-7$ by iterative method.** **QC 60045 MA 3251 APR 2022**

Let $f(x)=2x-\log_{10}x-7$

$f(1) = 2-\log_{10}1-7 = -5\ (-ve)$

$f(2) = 2\times2-\log_{10}2-7 = -03.3\ (-ve)$

$f(3) = 2\times3-\log_{10}3-7 = -1.4\ (-ve)$

$f(4) = 2\times4-\log_{10}4-7 = 0.39\ (+ve)$

Therefore, the root lies between 3 *and* 4.
Since $|f(4)|<|f(3)|$, the root is nearer to 4 and let the initial root be $x_0=3.7$.

Given equation can be written as

5. **Find the real root of $\cos x-2x+3=0$ correct to 3 decimal places using iteration method.** **QC 80221 MA 8491 MAY 2019**

Let $f(x)=\cos x-2x+3$

$f(0) = \cos 0-2(0)+3 = 4\ (+ve)$

$f(1) = \cos 1-2(1)+3 = 1.5\ (+ve)$

$f(2) = \cos 2-2(2)+3 = -1.4\ (-ve)$

Therefore, the root lies between 1 *and* 2.
Since $|f(2)|<|f(1)|$, the root is nearer to 2 and let the initial root be $x_0=1.6$.

Given equation can be written as

https://doi.org/10.5281/zenodo.15289885

$$2x-\log_{10}x-7=0$$
$$2x=\log_{10}x+7$$
$$x=\frac{1}{2}\left(\log_{10}x+7\right)$$
$$Let\ \phi(x)=\frac{1}{2}\left(\log_{10}x+7\right)$$
$$\therefore\ \phi'(x)=\frac{1}{2}\left(\frac{1}{x}\log_{10}e\right)=\frac{0.4343}{2x}=\frac{0.21715}{x}$$
$$\left|\phi'(x_0)\right|=\left|\phi'(3.7)\right|=\left|\frac{0.21715}{3.7}\right|<1$$

Therefore the successive roots are given by

$$x_1=\phi(x_0)=\phi(3.7)=\frac{1}{2}\left(\log_{10}3.7+7\right)=3.784$$
$$x_2=\phi(x_1)=\phi(3.784)=\frac{1}{2}\left(\log_{10}3.784+7\right)=3.7889$$
$$x_3=\phi(x_2)=\phi(3.7889)=\frac{1}{2}\left(\log_{10}3.7889+7\right)=3.7892$$
$$x_4=\phi(x_3)=\phi(3.789)=\frac{1}{2}\left(\log_{10}3.789+7\right)=3.7892$$

$\because\ x_3\approx x_4$, The root of the equation is $x=3.7892$

$$2x=\cos x+3$$
$$x=\frac{1}{2}\left(\cos x+3\right)$$
$$Let\ \phi(x)=\frac{1}{2}\left(\cos x+3\right)$$
$$\therefore\ \phi'(x)=\frac{1}{2}\left(-\sin x\right)$$
$$\left|\phi'(x_0)\right|=\left|\phi'(1.6)\right|=\left|-\frac{\sin 1.6}{1.6}\right|=\left|-\frac{0.9995}{1.6}\right|<1$$

Therefore the successive roots are given by

$$x_1=\phi(x_0)=\phi(1.5)=\frac{1}{2}\left(\cos 1.5+3\right)=1.5353$$
$$x_2=\phi(x_1)=\phi(1.5353)=\frac{1}{2}\left(\cos 1.5353+3\right)=1.5177$$
$$x_3=\phi(x_2)=\phi(1.5177)=\frac{1}{2}\left(\cos 1.5177+3\right)=1.5265$$
$$x_4=\phi(x_3)=\phi(1.265)=\frac{1}{2}\left(\cos 1.5265+3\right)=1.5221$$
$$x_5=\phi(x_4)=\phi(1.5221)=\frac{1}{2}\left(\cos 1.5221+3\right)=1.5243$$
$$x_6=\phi(x_5)=\phi(1.5243)=\frac{1}{2}\left(\cos 1.5243+3\right)=1.5232$$

$\because\ x_6\approx x_7$, The root of the equation is $x=1.5222$

.

6. Find the smallest positive root of $x^3-2x-5=0$ by fixed point iteration method?

QC 50785 MA 6459 DEC 2017

Let $f(x)=x^3-2x-5$

$$f(0)\ =\ 0^3-2(0)-5=\ -5\ \ (-ve)$$
$$f(1)\ =\ 1^3-2(1)-5=\ -6\ \ (-ve)$$
$$f(2)\ =\ 2^3-2(2)-5=\ -1\ (-ve)$$
$$f(3)\ =\ 3^3-2(3)-5=\ 16\ (+ve)$$

Therefore, the root lies between $2\ and\ 3$.

Since $\left|f(2)\right|<\left|f(3)\right|$, the root is nearer to 2 and let the initial root be $x_0=2.3$.

7. Find the positive root of $x^3+5x-3=0$ using fixed point iteration starting with 0.6 as first iteration

Given equation can be written as $x=\frac{1}{5}\left(3-x^2\right)$

$$\therefore\ \ \phi(x)=\frac{1}{5}\left(3-x^2\right)\text{ and}$$
$$\phi'(x)=-\frac{2x}{5}$$

Here $\left|\phi'(x_0)\right|=\left|\phi'(0.6)\right|=\left|-\frac{2}{5}(0.6)\right|<1$

$$\therefore\ \ x_1=\phi(x_0)\ =\ \frac{1}{5}\left(3-0.6^2\right)\ =\ 0.528$$

https://doi.org/10.5281/zenodo.15289885

Given equation can be written as

$$x^3 = 2x+5$$

$$x = (2x+5)^{\frac{1}{3}}$$

$$Let\ \phi(x) = (2x+5)^{\frac{1}{3}}$$

$$\therefore\ \phi'(x) = \frac{2}{3}(2x+5)^{-\frac{2}{3}}$$

$$|\phi'(x_0)| = |\phi'(2.3)| = \left|\frac{2}{3}(2\times2.3+5)^{-\frac{2}{3}}\right| = 0.13 < 1$$

Therefore the successive roots are given by

$$x_1 = \phi(x_0) = \phi(2.3) = (2\times2.3+5)^{\frac{1}{3}} = 2.1253$$

$$x_2 = \phi(x_1) = \phi(2.1253) = (2\times2.1253+5)^{\frac{1}{3}} = 2.0992$$

$$x_3 = \phi(x_2) = \phi(2.0992) = (2\times2.0992+5)^{\frac{1}{3}} = 2.0952$$

$$x_4 = \phi(x_3) = \phi(2.0952) = (2\times2.0952+5)^{\frac{1}{3}} = 2.0946$$

$$x_5 = \phi(x_4) = \phi(2.0956) = (2\times2.0956+5)^{\frac{1}{3}} = 2.0945$$

$$x_6 = \phi(x_5) = \phi(2.0945) = (2\times2.0945+5)^{\frac{1}{3}} = 2.0945$$

$\because\ x_5 \approx x_6$, The root of the equation is $x = 2.0945$

$$x_2 = \phi(x_1) = \frac{1}{5}\left(3-0.528^2\right) = 0.544$$

$$x_3 = \phi(x_2) = \frac{1}{5}\left(3-0.54^2\right) = 0.5416$$

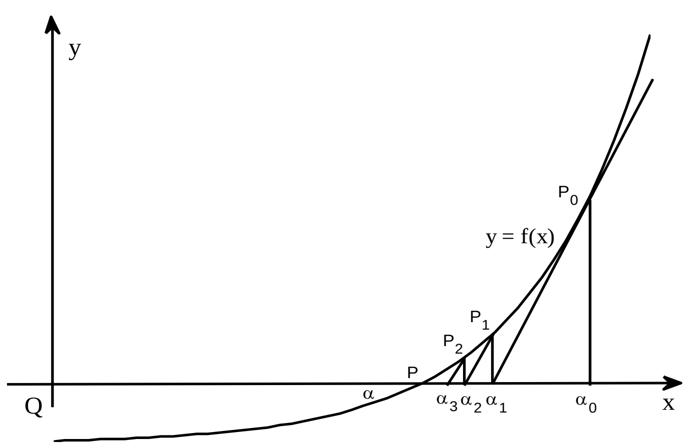
$$x_4 = \phi(x_3) = \frac{1}{5}\left(3-0.5416^2\right) = 0.5413$$

$\therefore$ the approximate root is x = 0.5413

..

Newton-Raphson Method

Geometrical Interpretation



https://doi.org/10.5281/zenodo.15289885

Let $x=\alpha$ be an exact root of $f(x)=0$. The curve cuts the $x-$axis at P whose coordinate is α.

Let α_0 be an approximate root of $f(x)=0$. The ordinate at $x=\alpha_0$ meets the curve at $P_0\left[\alpha_0,\ f\left(\alpha_0\right)\right]$.

The slope of the tangent at P_0 to the curve is $f'(\alpha_0)$.

The equation of tangent is $y-f(\alpha_0)=f'(\alpha_0)(x-\alpha_0)$

This cuts the x-axis at $x=\alpha_1$. To get the point, solve the equation of tangent with $y=0$

$$-f(\alpha_0)=f'(\alpha_0)(x-\alpha_0)$$

$$x=\alpha_1=\alpha_0-\frac{f(\alpha_0)}{f'(\alpha_0)}.$$

This α_1 is nearer to α than α_0. Continuing in this manner. We get a sequence $\alpha_0, \alpha_1, \alpha_2, ...$ a better approximation to the exact root α.

Results:

- The method is also called method of tangents
- Advantage of this method is that even complex root can also be calculated
- The error at any stage is proportional to the square of the error in the previous stage
- If $f'(x)$ is very small in the neighborhood of the root, the method may fail. In this case graph of $y=f(x)$ is almost parallel to $x-$axis at the root.
- This method is useful if $f'(x)$ is large in the neighborhood of the root. In this case the graph of $y=f(x)$ is nearly vertical near the root.

Newton-Raphson Method

(To find the root of algebraic or transcendental equation)

Working Rule:

- Let $f(x)=0$ be the given equation
- Choose two subsequent integers a & b such that $f(a)$ & $f(b)$ are of opposite signs. i.e. put $x=0, 1, 2, ...$, check the signs of $f(0), f(1), f(2), ...$ and find a & b.
- Assume that $f(a)=-ve$ & $f(b)=+ve$. Then the root lies between a & b.
- Then choose initial root $x_0\in(a,b)$. Hint: If $|f(a)|<|f(b)|$, then the root is nearer to 'a'.
- Then the successive roots are given by $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)},\ n=0, 1, ...$
- Continue this process till the required accuracy is reached.

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1. State the criterion for convergence of Newton-Raphson method.

QC 60045 MA 3251 APR 22

The convergence condition for NR method is

$|f(x).f''(x)| < [f'(x)]^2$

2. Write Newton-Raphson method for the solution of $f(x)=0$. QC 53250 MA 6452 MAY 2019

Newton Raphson formula to find the solution of

$f(x)=0$ is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, \ldots..$

3. Mention the order and condition for the convergence of Newton-Raphson method.

QC 57506 MA 6452 MAY 2016

The convergence condition for NR method is

$$|f(x).f''(x)| < [f'(x)]^2$$

The order of convergence of NR method is 2.

4. What are the merits of Newton-Raphson method? QC 50782 MA 6452 NOV 2017

- When $f'(x)$ is very large, this method will be useful.
- Since the order of convergence of NR method is more than the other methods, the convergence will be faster.

If approximate roots are known NR method will give more improved root

5. Find the iterative formula by Newton's Method for $\dfrac{1}{N}$, where N is a positive integer.

QC 20753 MA 6452 NOV 2018

Let $x = \dfrac{1}{N}$ i.e. $N = \dfrac{1}{x}$ i.e. $N - \dfrac{1}{x} = 0$

Let $f(x) = N - \dfrac{1}{x}$ and $f'(x) = \dfrac{1}{x^2}$

By Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(N - \dfrac{1}{x_n}\right)}{\left(\dfrac{1}{x_n^2}\right)}$$

$$= 2x_n - Nx_n^2$$

6. Derive a formula to find the value of $N^{\frac{1}{2}}$ where $N \neq 0$, using Newton-Raphson method.

QC 80610 MA 6452 NOV 2016

Let $x = \sqrt{N}$ i.e. $x^2 = N$

i.e. $x^2 - N = 0$

Let $f(x) = x^2 - N$ and $f'(x) = 2x$

By Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(x_n^2 - N\right)}{\left(2x_n\right)}$$

$$= x_n^2 + N$$

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7. Find the Newton Raphson formula to find $\sqrt[3]{N}$ where N is a positive integer.

QC 80221 MA 8491 MAY 2019

Let $x = \sqrt[3]{N} = N^{\frac{1}{3}}$

i.e. $x^3 = N$

i.e. $x^3 - N = 0$

Let $f(x) = x^3 - N$ and $f'(x) = 3x^2$

By Newton's formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{\left(x_n^3 - N\right)}{\left(3x_n^2\right)}$$

$$= \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$= \frac{2x_n^3 + N}{3x_n^2}$$

8. Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method. **QC 20817 MA 8452 APR 2022**

Let $f(x) = x^4 - x - 10$ then $f'(x) = 4x^3 - 1$

$f(1) = 1^4 - 1 - 10 = -10$

$f(2) = 2^4 - 1 - 10 = 5$

Therefore, the root lies between 1 *and* 2.
Since $|f(2)| < |f(1)|$, the root is nearer to 2 and let the initial root be $x_0 = 1.7$.

By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,\ldots\ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.7 - \frac{(1.7)^4 - 1.7 - 10}{4(1.7)^3 - 1}$$

$$= 1.7 - \frac{-3.34}{18.652} = 1.879$$

9. Find the real positive root $3x - \cos x - 1 = 0$ of by Newton Raphson method, correct to 3 decimal places. **QC 53250 MA 6452 MAY 2019**

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

$f(0) = -2 \quad (-ve)$

$f(1) = 1.4597 \quad (+ve)$

Therefore, the root lies between 0 *and* 1. Since $|f(1)| < |f(0)|$, the root is nearer to 1 and let the initial root be $x_0 = 0.6$.

By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,\ldots\ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.6 - \frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)}$$

$$= 0.6 - \frac{-0.02534}{3.56464}$$

$$= 0.60711$$

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$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.879 - \frac{(1.879)^4 - 1.879 - 10}{4(1.879)^3 - 1}$$

$$= 1.879 - \frac{0.5864}{25.536} = 1.856$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{-0.0102}{24.57} = 1.856$$

$\because\ x_2 \approx x_3$, The root of the equation is $x = 1.856$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)}$$

$$= 0.6 - \frac{-0.00001}{3.57049}$$

$$= 0.6071$$

$\because\ x_1 \approx x_2$, The root of the equation is $x = 0.6071$

10. Find the smallest positive root of $x^3 - 2x + 0.5 = 0$. **QC 72071 MA 6452 MAY 2017**

Let $f(x) = x^3 - 2x + 0.5$ and $f'(x) = 3x^2 - 2$

$$f(0) = 0^3 - 2(0) + 0.5 = 0.5$$

$$f(1) = 1^3 - 2(1) + 0.5 = -0.5$$

Therefore the root lies between 0 and 1 and let $x_0 = 0.5$. By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, 3, \ldots\ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{(0.5)^3 - 2(0.5) + 0.5}{3(0.5)^2 - 2}$$

$$= 0.5 - \frac{-0.375}{-1.25}$$

$$= 0.2$$

11. Find the positive root of $x^4 - x - 9 = 0$ **by Newton method. QC 27331 MA 6452 NOV 2015**

Let $f(x) = x^4 - x - 9$ then $f'(x) = 4x^3 - 1$

$$f(1) = 1^4 - 1 - 9 = -9$$

$$f(2) = 2^4 - 1 - 9 = 6$$

Therefore, the root lies between 1 *and* 2.
Since $|f(2)| < |f(1)|$, the root is nearer to 2 and let the initial root be $x_0 = 1.7$. By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, 3, \ldots\ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.7 - \frac{(1.7)^4 - 1.7 - 9}{4(1.7)^3 - 1}$$

$$= 1.7 - \frac{-2.34}{18.652}$$

$$= 1.825$$

https://doi.org/10.5281/zenodo.15289885

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.2 - \frac{(0.2)^3 - 2(0.2) + 0.2}{3(0.2)^2 - 2}$$

$$= 0.2 - \frac{-0.192}{-1.88}$$

$$= 0.0978$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.0978 - \frac{(0.0978)^3 - 2(0.0978) + 0.2}{3(0.0978)^2 - 2}$$

$$= 0.0978 - \frac{0.0053375}{-1.9713}$$

$$= 0.1005$$

$\because\ x_2 \approx x_3$, The root of the equation is $x = 0.1005$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.825 - \frac{(1.825)^4 - 1.825 - 9}{4(1.825)^3 - 1}$$

$$= 1.825 - \frac{0.268}{23.313}$$

$$= 1.813$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.813 - \frac{(1.813)^4 - 1.813 - 9}{4(1.813)^3 - 1}$$

$$= 1.813 - \frac{-0.0088}{23.84}$$

$$= 1.833$$

$\because\ x_2 \approx x_3$, The root of the equation is $x = 1.813$

12. Find the positive real root of $x\log_{10} x = 1.2$ by Newton method. QC 90346 MA 8491 DEC 19

$f(x) = x\log_{10} x - 1.2$ then

$$f'(x) = x.\frac{1}{x}\log_{10} e + \log_{10} x = 0.4343 + \log_{10} x$$

$$f(1) = 1.\log_{10} 1 - 1.2 = -1.2\ (-ve)$$

$$f(2) = 2.\log_{10} 2 - 1.2 = -0.59\ (-ve)$$

$$f(3) = 3.\log_{10} 3 - 1.2 = 0.23\ (+ve)$$

Therefore, the root lies between 2 *and* 3.
Since $|f(3)| < |f(2)|$, the root is nearer to 3 and let the initial root be $x_0 = 2.7$.

By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\ ,\ n = 0,1,2,3,\ldots\ldots$$

13. Find the root of $4x - e^x = 0$ that lies between 2 and 3 by NR method.
QC 27335 MA 6459 DEC 2015

$f(x) = 4x - e^x$ then

$$f'(x) = 4 - e^x$$

$$f(2) = 4\times(2) - e^2 = 0.61\ (+ve)$$

$$f(3) = 4\times(3) - e^3 = -8.08\ (-ve)$$

Therefore, the root lies between 2 *and* 3.
Since $|f(2)| < |f(3)|$, the root is nearer to 2 and let the initial root be $x_0 = 2.3$.

By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\ ,\ n = 0,1,2,3,\ldots\ldots$$

https://doi.org/10.5281/zenodo.15289885

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.7 - \frac{2.7 \log_{10} 2.7 - 1.2}{0.4343 + \log_{10} 2.7}$$

$$= 2.7 - \frac{-0.03531}{0.86566} = 2.7407$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7407 - \frac{2.7407 \log_{10} 2.7407 - 1.2}{0.4343 + \log_{10} 2.7407}$$

$$= 2.7407 - \frac{0.000047012}{0.87216}$$

$$= 2.7406$$

Since $x_1 \approx x_2$, the root of the equation is $x = 2.7406$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.3 - \frac{4 \times (2.3) - e^{2.3}}{4 - e^{2.3}}$$

$$= 2.3 - \frac{-0.077418}{-5.97418}$$

$$= 2.287$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.287 - \frac{4 \times (2.287) - e^{2.287}}{4 - e^{2.287}}$$

$$= 2.287 - \frac{-0.69735}{-5.84535}$$

$$= 2.18$$

Since $x_1 \approx x_2$, the root of the equation is $x = 2.18$

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Solution of Simultaneous Linear Algebraic Equations

Direct Methods

1. Gauss Elimination Method 2. Gauss Jordan Method

- Gauss elimination method fails when any one of the pivots is zero or it is a very small number.
- In such situation, rearrange the equations in a different order to avoid zero pivot. Change the order of equations is called pivoting. In complete pivoting we interchange rows as well as columns, such that the largest element in the matrix of the system becomes the pivot element.
- Partial pivoting: During the elimination process, if $a_{ii} = 0$, then the $i-th$ column elements are searched for the numerically largest element. Let the $j-th$ row $(j > i)$ contains this element. Then interchange $i-th$ row with the $j-th$ row and proceed for elimination
- Basic principle of Gauss elimination method, the coefficient matrix is transformed to upper triangular matrix. For this one of the pivot element must be non zero.

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Elementary transformations

i. Interchange of $i-th$ and $j-th$ row (column)

ii. Multiply all elements of $i-th$ rwo by a number k

iii. Adding the elements of $i-th$ row to the corresponding elements in the $j-th$ row

Gauss Elimination Method (To solve the system of simultaneous equations)
Working Rule: • Consider a set of simultaneous equations in 3 variables, say, $a_1x+b_1y+c_1z=d_1;\ a_2x+b_2y+c_2z=d_2;\ a_3x+b_3y+c_3z=d_3$ • Form the augmented matrix $(A,B)=\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$. • Reduce the coefficient matrix A into upper triangular matrix by using elementary row operations. • Then by back substitution, solution is obtained.

1. Solve the equations $5x-2y=1,\ 4x+28y=23$ **using the Gauss elimination method.**

QC 80610 MA 6452 NOV 2016

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix} \boxed{5} & -2 & 1 \\ 4 & 28 & 23 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & -2 & 1 \\ 0 & 148 & 111 \end{pmatrix} R_2 \to 5R_2-4R_1 \quad \{(20\ 140\ 115)-(20\ -8\ 4)\}$$

By back substitution,

$$148y=111 \qquad 5x-2y=1$$

$$i.e.\ y=\frac{111}{148} \qquad 5x-\frac{111}{74}=1$$

$$x=\frac{1}{5}\left[1+\frac{111}{74}\right]=\frac{37}{74}$$

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2. Solve the following by Gauss Elimination method $10x + y = 18.141; \quad x + 10y = 28.14$

QC 72071 MA 6452 MAY 2017

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{1} & 10 & 28.14 \\ 10 & 1 & 18.14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 10 & 28.14 \\ 0 & -99 & -263.26 \end{pmatrix} R_2 \to R_2 - 10R_1 \quad \{(10\ 1\ 18.14) - (10\ 100\ 281.4)\}$$

By back substitution,

$$-99y = -263.26 \qquad x + 10y = 28.14$$

$$i.e. \quad y = 2.6592 \qquad x + 26.592 = 28.14$$

$$x = 1.548$$

3. Solve the system of equations by Gauss elimination method
$x + 2y + z = 3; \quad 2x + 3y + 3z = 10; \quad 3x - y + 2z = 13$ **QC 53250 MA 6452 MAY 2019**

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{1} & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & \boxed{-1} & 1 & 4 \\ 0 & -7 & -1 & 4 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \quad \{(2\ 3\ 3\ 10) - (2\ 4\ 2\ 6)\} \\ R_3 \to R_3 - 3R_1 \quad \{(3\ -1\ 2\ 13) - (3\ 6\ 3\ 9)\} \end{matrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & \boxed{-1} & 1 & 4 \\ 0 & 0 & -8 & -24 \end{pmatrix} R_3 \to R_3 - 7R_2 \ \{(0\ -7\ -1\ 4) - (0\ -7\ 7\ 28)\}$$

By back substitution, the solution is

$$\begin{aligned} -8z &= -24 & i.e.\ z &= 3 \\ -y + z &= 4 & i.e.\ y &= z - 4 = -1 \\ x + 2y + z &= 3 & i.e.\ x &= 3 - 2y - z = 2 \end{aligned}$$

4. Solve the following system of equations by Gauss Elimination Method.
$2y - z = -5; \quad x + 4y - 7z + t = -8; \quad 2x - y - t = -4; \quad x + y + z = 6$ **QC 41313 MA 6452 MAY 2018**

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{1} & 4 & -7 & 1 & -8 \\ 0 & 2 & -1 & 0 & -5 \\ 2 & -1 & 0 & -1 & -4 \\ 1 & 1 & 1 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & -7 & 1 & -8 \\ 0 & \boxed{2} & -1 & 0 & -5 \\ 0 & -9 & 14 & -3 & 12 \\ 0 & -3 & 8 & -1 & 14 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_1 \quad \{(2\ -1\ 0\ -1\ -4) - (2\ 8\ -14\ 2\ -16)\} \\ R_4 \to R_4 - R_1 \quad \{(1\ 1\ 1\ 0\ 6) - (1\ 4\ -7\ 1\ -8)\} \end{matrix}$$

$$= \begin{pmatrix} 1 & 4 & -7 & 1 & -8 \\ 0 & 2 & -1 & 0 & -5 \\ 0 & 0 & \boxed{19} & -6 & -21 \\ 0 & 0 & 13 & -2 & 13 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to 2R_3 + 9R_2 \ \{(0\ -18\ 24\ -6\ 24) + (0\ 18\ -9\ 0\ -45)\} \\ R_4 \to 2R_4 + 3R_2 \ \{(0\ -6\ 16\ -2\ 28) + (0\ 6\ -3\ 0\ -15)\} \end{matrix}$$

$$= \begin{pmatrix} 1 & 4 & -7 & 1 & -8 \\ 0 & 2 & -1 & 0 & -5 \\ 0 & 0 & 19 & -6 & -21 \\ 0 & 0 & 0 & 40 & 520 \end{pmatrix} \begin{matrix} \\ \\ \\ R_4 \to 19R_4 - 13R_3 \{(0\ 0\ 247\ -38\ 247) - (0\ 0\ 247\ -78\ -273)\} \end{matrix}$$

By back substitution, we have

$40t = 520$	$19z - 6t = -21$	$2y - z = -5$	$x + 4y - 7z + t = -8$
$t = 13$	$19z - 78 = -21$	$2y = 3 - 5$	$x - 4 - 21 + 13 = -8$
	$19z = 57$	$y = -1$	$x = 4$
	$z = 3$		

5. Solve, by Gauss elimination with partial pivoting method, the system of following equations correct to 3 decimal places. $2x + y + z = 10, \ 3x + 2y + 3z = 18, \ x + 4y + 9z = 16.$

QC 20753 MA 6452 NOV 2018

The augmented matrix of the given system is

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$$(A,B)=\begin{pmatrix} \boxed{1} & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & \boxed{-7} & -17 & -22 \\ 0 & -10 & -24 & -30 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \quad \{(2\ 1\ 1\ 10)-(2\ 8\ 18\ 32)\} \\ R_3 \to R_3 - 3R_1 \ \{(3\ \ 2\ \ 3\ 18)-(3\ 12\ \ 27\ \ 48)\} \end{matrix}$$

$$=\begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & \boxed{-7} & -17 & -22 \\ 0 & 0 & 2 & 10 \end{pmatrix} R_3 \to 7R_3 - 10R_2 \{(0 -70 -168 -210)-(0 -70 -170 -220)\}$$

By back substitution, we have

$$2z = 10 \qquad -7y - 17z = -22 \qquad x + 4y + 9z = 16$$

$$z = 5 \qquad -7y - 85 = -22 \qquad x - 36 + 45 = 16$$

$$7y = -63 \qquad x = 7$$

$$y = -9$$

6. Solve the equations by Gauss elimination method:

$2x + y + 4z = 12; \quad 8x - 3y + 2z = 20; \quad 4x + 11y - z = 33$ **QC 57506 MA 6452 MAY 2016**

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix} \boxed{2} & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & \boxed{-7} & -14 & -28 \\ 0 & 9 & -9 & 9 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \quad \{(8\ -3\ 2\ 20)-(8\ 4\ 16\ 48)\} \\ R_3 \to R_3 - 2R_1 \ \{(4\ \ 11\ \ -1\ 33)-(4\ 2\ 8\ 24)\} \end{matrix}$$

$$=\begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -189 & -189 \end{pmatrix} R_3 \to 7R_3 + 9R_2 \{(0\ \ 63\ \ -63\ \ 63)+(0\ \ -63\ \ -126\ \ -252)\}$$

By back substitution, the solution is given by

$$-189z = -189 \quad \text{i.e.} \quad z = 1$$

$$-7y - 14z = -28$$
$$-7y = -28 + 14$$
$$y = 2$$

$$2x + y + 4z = 12$$
$$2x + 2 + 4 = 12$$
$$2x = 6$$
$$x = 3$$

7. Solve, by Gauss elimination with partial pivoting method, the system of equations

$2x + y + z = 10; \quad 3x + 2y + 3z = 18; \quad x + 4y + 9z = 16$ **QC 20753 MA 6452 DEC 2018**

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{2} & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & \boxed{1} & 3 & 6 \\ 0 & 7 & 17 & 22 \end{pmatrix} \begin{matrix} \\ R_2 \to 2R_2 - 3R_1 \quad \{(6\ 4\ 6\ 36) - (6\ 3\ 3\ 30)\} \\ R_3 \to 2R_3 - R_1 \quad \{(2\ 8\ 18\ 32) - (2\ 1\ 1\ 10)\} \end{matrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{pmatrix} R_3 \to R_3 - 7R_2 \ \{(0\ 7\ 17\ 22) - (0\ 7\ 21\ 42)\}$$

By back substitution, the solution is given by

$$-4z = -20$$
$$z = 5$$

$$y + 3z = 6$$
$$y + 15 = 6$$
$$y = -9$$

$$2x + y + z = 10$$
$$2x - 9 + 5 = 10$$
$$2x = 10 + 4$$
$$x = 7$$

Gauss Jordan Method (To solve the system of simultaneous equations)
Working Rule: • Consider a set of simultaneous equations in 3 variables, say, $a_1x+b_1y+c_1z=d_1;\ a_2x+b_2y+c_2z=d_2;\ a_3x+b_3y+c_3z=d_3$ • Form the augmented matrix $(A,B)=\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$. • Reduce the coefficient matrix A into diagonal(or unit) matrix by using elementary row operations. • Then directly, solution is obtained.

1. **What is the procedure of Gauss-Jordan method?** **QC 57506 MA 6452 MAY 2016**

- Write the augmented matrix of the given system
- The elements above and below the pivot element are converted to zero
- Thus the augmented matrix is converted into a diagonal matrix
- The solution is got directly without back-substitution

2. **Compare Gauss Elimination method and Gauss Jordan method for solving a linear system.** **QC 80221 MA 8491 MAY 2019**

Gauss Elimination Method	Gauss Jordan Method
Matrix is transformed in to upper triangular matrix	Matrix is transformed in to unit matrix
Convenient for less number of equations	Convenient for more number of equations
Back substitution to the reduced matrix gives the solution	The vector B in the reduced matrix gives the solution

3 **Solve** $3x+2y=4;\quad 2x-3y=7$ **by Gauss Jordan method.** **QC 90346 MA 8491 DEC 2019**

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix} \boxed{3} & 2 & 4 \\ 2 & -3 & 7 \end{pmatrix}$$

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$$= \begin{pmatrix} 3 & 2 & 4 \\ 0 & \boxed{-13} & 13 \end{pmatrix} R_2 \to 3R_2 - 2R_1 \quad \{(6\ \ -9\ \ 21) - (6\ \ 4\ \ 8)\}$$

$$= \begin{pmatrix} 39 & 0 & 78 \\ 0 & -13 & 13 \end{pmatrix} R_1 \to 13R_1 + 2R_2 \quad \{(39\ \ 26\ \ 52) + (0\ \ -26\ \ 26)\}$$

By back substitution, we have

$$-13y = 13 \quad i.e. \ \ y = -1$$

$$39x = 78 \quad i.e. \quad x = 2$$

4. Solve the following system of equations by Gauss Jordan method.

$2x + y + 4z = 12; \quad 8x - 3y + 2z = 20; \quad 4x + 11y - z = 33$ **QC 41313 MA 6452 MAY 2018**

The augmented matrix of the given system is

$$(A, B) = \begin{pmatrix} \boxed{2} & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & \boxed{-7} & -14 & -28 \\ 0 & 9 & -9 & 9 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - 4R_1 \quad \{(8\ \ -3\ \ 2\ \ 20) - (8\ \ 4\ \ 16\ \ 48)\} \\ R_3 \to R_3 - 2R_1 \quad \{(4\ \ 11\ \ -1\ \ 33) - (4\ \ 2\ \ 8\ \ 24)\} \end{matrix}$$

$$= \begin{pmatrix} 14 & 0 & 14 & 56 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & \boxed{-189} & -189 \end{pmatrix} \begin{matrix} R_1 \to 7R_1 + R_2 \quad \{(14\ \ 7\ \ 28\ \ 84) + (0\ \ -7\ \ -14\ \ -28)\} \\ \\ R_3 \to 7R_3 + 9R_2 \quad \{(0\ \ 63\ \ -63\ \ 63) + (0\ \ -63\ \ -126\ \ -252)\} \end{matrix}$$

$$= \begin{pmatrix} 2646 & 0 & 0 & 7938 \\ 0 & -1323 & 0 & -2646 \\ 0 & 0 & -189 & -189 \end{pmatrix} \begin{matrix} R_1 \to 189R_1 + 14R_3 \{(2646\ \ 0\ \ 2646\ \ 10584) + (0\ \ 0\ \ -2646\ \ -2646)\} \\ R_2 \to 189R_2 - 12R_3 \{(0\ \ -1323\ \ -2646\ \ -5292) - (0\ \ 0\ \ -2646\ \ -2646)\} \\ \\ \end{matrix}$$

$$-189z = -189 \quad i.e. \quad z = 1$$

By back substitution, the solution is given by $-1323y = -2646 \ \ i.e. \quad y = 2$

$$2646x = 7938 \quad i.e. \quad x = 3$$

5. Solve the following system of equations by Gauss-Jordan method:

$x - y + z = 1;\ -3x + 2y - 3z = -6;\ 2x - 5y + 4z = 5$ **QC 60045 MA 3251 APR 2022**

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{1} & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & \boxed{-1} & 0 & -3 \\ 0 & -3 & 2 & 3 \end{pmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 & \{(-3\ 2\ -3\ -6) + (3\ -3\ 3\ 3)\} \\ R_3 \to R_3 - 2R_1 & \{(2\ -5\ 4\ 5) - (2\ -2\ 2\ 2)\} \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & \boxed{2} & 12 \end{pmatrix} \begin{matrix} R_1 \to R_1 - R_2 & \{(1\ -1\ 1\ 1) + (0\ -1\ 0\ -3)\} \\ \\ R_3 \to R_3 - 3R_2 & \{(0\ -3\ 2\ 3) - (0\ -3\ 0\ -9)\} \end{matrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & -4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{pmatrix} \begin{matrix} R_1 \to 2R_1 - R_3 & \{(2\ 0\ 2\ 8) + (0\ 0\ 2\ 12)\} \\ & R_2 \to R_2 \end{matrix}$$

$2x = -4,$

By back substitution, $-y = -3,$

$2z = 12$

Hence, from third equation $z = 6$, from second equation $y = 3$, **f**rom first equation $x = -2$

6. Using Gauss-Jordan method, solve the system of equations

$x + y + z = 9;\ 2x - 3y + 4z = 13;\ 3x + 4y + 5z = 40$ **QC 20817 MA 8452 APR 2022**

The augmented matrix of the given system is

$$(A,B) = \begin{pmatrix} \boxed{1} & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & \boxed{-5} & 2 & -5 \\ 0 & 1 & 2 & 13 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 & \{(2\ -3\ 4\ 13) + (-2\ -2\ -2\ -18)\} \\ R_3 \to R_3 - 3R_1 & \{(3\ 4\ 5\ 40) + (-3\ -3\ -3\ -27)\} \end{matrix}$$

$$=\begin{pmatrix} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \boxed{12} & 60 \end{pmatrix} \begin{matrix} R_1 \to 5R_1 + R_2 \ \{(5\ 5\ 5\ 45) + (0\ -5\ 2\ -5)\} \\ \\ R_3 \to 5R_3 + R_2 \ \{(0\ 5\ 10\ 65) + (0\ -5\ 2\ -5)\} \end{matrix}$$

$$=\begin{pmatrix} 60 & 0 & 0 & 60 \\ 0 & -30 & 0 & -90 \\ 0 & 0 & 12 & 60 \end{pmatrix} \begin{matrix} R_1 \to 12R_1 - 7R_3 \ \{(60\ 0\ 84\ 480) - (0\ 0\ 84\ 420)\} \\ R_2 \to 6R_2 - R_3 \ \{(0\ -30\ 12\ -30) - (0\ 0\ 12\ 60)\} \\ \end{matrix}$$

$60x = 60,$

By back substitution, $-30y = -90,$

$12z = 60.$

Hence, from third equation $z = 5$, from second equation $y = 3$, from first equation $x = 1$

7. Solve the system of the following equations using Gauss Jordan method correct to two decimal places. $2x_1 + 2x_2 - x_3 + x_4 = 4,\ 4x_1 + 3x_2 - x_3 + 2x_4 = 6,\ 8x_1 + 5x_2 - 3x_3 + 4x_4 = 12,$ $3x_1 + 3x_2 - 2x_3 + 2x_4 = 6$ **QC 72071 MA 6452 MAY 2017**

The augmented matrix of the given system is

$$(A, B) = \begin{pmatrix} \boxed{2} & 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 2 & 6 \\ 8 & 5 & -3 & 4 & 12 \\ 3 & 3 & -2 & 2 & 6 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 2 & -1 & 1 & 4 \\ 0 & \boxed{-1} & 1 & 0 & -2 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \ \{(4\ 3\ -1\ 2\ 6) - (4\ 4\ -2\ 2\ 8)\} \\ R_3 \to R_3 - 4R_1 \ \{(8\ 5\ -3\ 4\ 12) - (8\ 8\ -4\ 4\ 16)\} \\ R_4 \to 2R_4 - 3R_1 \ \{(6\ 6\ -4\ 4\ 12) - (6\ 6\ -3\ 3\ 12)\} \end{matrix}$$

$$=\begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & \boxed{-2} & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 2R_2 \ \{(2\ 2\ -1\ 1\ 4) + (0\ -2\ 2\ 0\ -4)\} \\ \\ R_3 \to R_3 - 3R_2 \ \{(0\ -3\ 1\ 0\ -4) - (0\ -3\ 3\ 0\ -6)\} \\ R_4 \to R_4 \end{matrix}$$

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$$= \begin{pmatrix} 4 & 0 & 0 & 2 & 2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & \boxed{-2} & 0 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{matrix} R_1 \to 2R_1 + R_3 \{(4\ 0\ 2\ \ 2\ 0) + (0\ \ 0\ \ -2\ \ 0\ \ 2)\} \\ R_2 \to 2R_2 + R_3 \{(0\ -2\ 2\ \ 0\ -4) + (0\ \ 0\ \ -2\ \ 0\ \ 2)\} \\ \\ R_4 \to 2R_4 - R_3 \{(0\ 0\ -2\ \ 2\ 0) - (0\ \ 0\ \ -2\ \ 0\ \ 2)\} \end{matrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 & 0 & 4 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 0 & \boxed{-2} & 0 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix} \begin{matrix} R_1 \to R_1 - R_4 \{(4\ 0\ 0\ \ 2\ 2) - (0\ \ 0\ \ 0\ \ 2\ \ -2)\} \\ R_2 \to R_2 \\ R_3 \to R_3 \\ \end{matrix}$$

By back substitution, the solution is

$2x_4 = -2$ *i.e.* $x_4 = -1$

$-2x_3 = 2$ *i.e.* $x_3 = -1$

$-2x_2 = -2$ *i.e.* $x_2 = 1$

$4x_1 = 4$ *i.e.* $x_1 = 1$

Gauss Jordan Method (To find the inverse of a matrix A)

Working Rule:

- Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a given matrix.
- Write the augmented matrix $(A, B) = \begin{pmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{pmatrix}$.
- Reduce the coefficient matrix A into unit matrix by using elementary row operations and it becomes $(A, B) \approx \begin{pmatrix} 1 & 0 & 0 & j & k & l \\ 0 & 1 & 0 & m & n & o \\ 0 & 0 & 1 & p & q & r \end{pmatrix}$.
- In the reduced matrix, inverse of the matrix is given by $A^{-1} = \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}$.

Note: Inverse of the matrix exists, if and only if it is a non singular square matrix.

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Elementary Operations:

- Interchange of any two rows
- Multiplication of elements of any row by a non zero constant
- Addition of elements of any row with the elements of other row

1. Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ using Gauss Jordan method.

QC 72071 MA 6452 MAY 2017

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix} \boxed{2} & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 3 & -3 & 2 & 0 \\ 0 & 7 & 17 & -1 & 0 & 2 \end{pmatrix}\begin{matrix} \\ R_2 \to 2R_2-3R_1 \quad \{(6\ 4\ 6\ 0\ 2\ 0)-(6\ 3\ 3\ 3\ 0\ 0)\} \\ R_3 \to 2R_3-R_1 \quad \{(2\ 8\ 18\ 0\ 0\ 2)-(2\ 1\ 1\ 1\ 0\ 0)\} \end{matrix}$$

$$=\begin{pmatrix} 2 & 0 & -2 & 4 & -2 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & \boxed{-4} & 20 & -14 & 2 \end{pmatrix}\begin{matrix} R_1 \to R_1-R_2 \quad \{(2\ 1\ 1\ 1\ 0\ 0)-(0\ 1\ 3\ -3\ 2\ 0)\} \\ \\ R_3 \to R_3-7R_2 \quad \{(0\ -7\ 17\ -1\ 0\ 2)-(0\ 7\ 21\ -21\ 14\ 0)\} \end{matrix}$$

$$=\begin{pmatrix} 4 & 0 & 0 & -12 & 10 & -2 \\ 0 & 4 & 0 & 48 & -34 & 6 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{pmatrix}\begin{matrix} R_1 \to 2R_1-R_3 \ \{(4\ 0\ -4\ 8\ -4\ 0)-(0\ 0\ -4\ 20\ -14\ 2)\} \\ R_2 \to 4R_2+3R_3 \ \{(0\ 4\ 12\ -12\ 8\ 0)+(0\ 0\ -12\ 60\ -42\ 6)\} \\ \\ \end{matrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & -3 & \frac{10}{4} & \frac{-2}{4} \\ 0 & 1 & 0 & 12 & -\frac{34}{4} & \frac{6}{4} \\ 0 & 0 & 1 & -5 & \frac{-14}{-4} & \frac{2}{-4} \end{pmatrix}\begin{matrix} R_1 \to \frac{R_1}{4} \\ R_2 \to \frac{R_2}{4} \\ R_3 \to \frac{R_3}{-4} \end{matrix}$$

Hence $$A^{-1}=\begin{pmatrix} -3 & \frac{10}{4} & -\frac{2}{4} \\ 12 & -\frac{34}{4} & \frac{6}{4} \\ -5 & \frac{-14}{-4} & \frac{2}{-4} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} -12 & 10 & -2 \\ 48 & -34 & 6 \\ -20 & 14 & -2 \end{pmatrix}$$

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2. **Using Gauss-Jordan method, find the inverse of** $A=\begin{pmatrix}4 & 1 & 2\\ 2 & 3 & -1\\ 1 & -2 & 2\end{pmatrix}$ **QC 50782 MA 6452 NOV 2017**

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix}\boxed{4} & 1 & 2 & 1 & 0 & 0\\ 2 & 3 & -1 & 0 & 1 & 0\\ 1 & -2 & 2 & 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}4 & 1 & 2 & 1 & 0 & 0\\ 0 & \boxed{5} & -4 & -1 & 2 & 0\\ 0 & -9 & 6 & -1 & 0 & 4\end{pmatrix}\begin{matrix}\\ R_2\to 2R_2-R_1 \quad \{(4\ 6\ -2\ 0\ 2\ 0)-(4\ 1\ 2\ 1\ 0\ 0)\}\\ R_3\to 4R_3-R_1 \quad \{(4\ -8\ 8\ 0\ 0\ 4)-(4\ 1\ 2\ 1\ 0\ 0)\}\end{matrix}$$

$$=\begin{pmatrix}20 & 0 & 14 & 6 & -2 & 0\\ 0 & 5 & -4 & -1 & 2 & 0\\ 0 & 0 & \boxed{-6} & -14 & 18 & 20\end{pmatrix}\begin{matrix}R_1\to 5R_1-R_2 \quad \{(20\ 5\ 10\ 5\ 0\ 0)-(0\ 5\ -4\ -1\ 2\ 0)\}\\ \\ R_3\to 5R_3+9R_2 \quad \{(0\ -45\ 30\ -5\ 0\ 20)+(0\ 45\ -36\ -9\ 18\ 0)\}\end{matrix}$$

$$=\begin{pmatrix}120 & 0 & 0 & -160 & 240 & 280\\ 0 & 30 & 0 & 50 & -60 & -80\\ 0 & 0 & -6 & -14 & 18 & 20\end{pmatrix}\begin{matrix}R_1\to 6R_1+14R_3\ \{(120\ 0\ 84\ 36\ -12\ 0)+(0\ 0\ -84\ -196\ 252\ 280)\}\\ R_2\to 6R_2-4R_3\ \{(0\ 30\ -24\ -6\ 12\ 0)-(0\ 0\ -24\ -56\ 72\ 80)\}\\ \\ \end{matrix}$$

$$=\begin{pmatrix}1 & 0 & 0 & \frac{-160}{120} & 2 & \frac{280}{120}\\ 0 & 1 & 0 & \frac{50}{30} & -2 & \frac{-80}{30}\\ 0 & 0 & 1 & \frac{-14}{-6} & -3 & \frac{20}{-6}\end{pmatrix}\begin{matrix}R_1\to \frac{R_1}{120}\\ R_2\to \frac{R_2}{30}\\ R_3\to \frac{R_3}{-6}\end{matrix}$$

Hence $$A^{-1}=\begin{pmatrix}\frac{-160}{120} & 2 & \frac{280}{120}\\ \frac{50}{30} & -2 & \frac{-80}{30}\\ \frac{-14}{-6} & -3 & \frac{20}{-6}\end{pmatrix}=\begin{pmatrix}\frac{-4}{3} & 2 & \frac{7}{3}\\ \frac{5}{3} & -2 & \frac{-8}{3}\\ \frac{7}{3} & -3 & \frac{10}{-3}\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-4 & 6 & 7\\ 5 & -6 & -8\\ 7 & -9 & -10\end{pmatrix}$$

https://doi.org/10.5281/zenodo.15289885

3. Consider the system of equations of the form $AX = B$, where $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ -3 \\ 7 \end{pmatrix}$. Find by using Gauss Jordan method, (i) A^{-1} (ii) the solution of the given system. **QC 41316 MA 6459 MAY 2018**

(i) Refer previous example for A^{-1}

(ii) To find the solution of the system $AX = B$

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{pmatrix}\begin{pmatrix} 7 \\ -3 \\ 7 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -28-18+49 \\ 35+18-56 \\ 49+27-70 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

4. Using Gauss Jordan method, find the inverse of $A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ **QC 20753 MA 6452 NOV 2018**

The augmented matrix of the given system is

$$(A, B) = \begin{pmatrix} \boxed{3} & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 5 & 1 & 0 \\ 0 & -1 & 1 & -5 & 0 & 3 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 + 5R_1 \quad \{(-15\ 6\ -5\ 0\ 1\ 0) - (15\ -5\ 5\ 5\ 0\ 0)\} \\ R_3 \to 3R_3 - 5R_1 \quad \{(15\ -6\ 6\ 0\ 0\ 3) - (15\ -5\ 5\ 5\ 0\ 0)\} \end{matrix}$$

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$$=\begin{pmatrix}3 & 0 & 1 & 6 & 1 & 0\\ 0 & 1 & 0 & 5 & 1 & 0\\ 0 & 0 & \boxed{1} & 0 & 1 & 3\end{pmatrix}\begin{matrix}R_1\to R_1+R_2 \quad \{(3\ -1\ 1\ 1\ 0\ 0)-(0\ 1\ 0\ 5\ 1\ 0)\}\\ \\ R_3\to R_3+R_2 \quad \{(0\ -1\ 1\ -5\ 0\ 3)-(0\ 1\ 0\ 5\ 1\ 0)\}\end{matrix}$$

$$=\begin{pmatrix}3 & 0 & 0 & 6 & 0 & -3\\ 0 & 1 & 0 & 5 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 3\end{pmatrix}\begin{matrix}R_1\to R_1-R_3 \quad \{(3\ 0\ 1\ 6\ 1\ 0)-(0\ 0\ 1\ 0\ 1\ 3)\}\\ R_2\to R_2\\ \end{matrix}$$

$$=\begin{pmatrix}1 & 0 & 0 & 2 & 0 & -1\\ 0 & 1 & 0 & 5 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 3\end{pmatrix} R_1\to \frac{R_1}{3}$$

Hence $A^{-1}=\begin{pmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{pmatrix}$

5. Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$

QC 80610 MA 6452 NOV 2016

The augmented matrix of the given system is

$$(A,B)=\begin{pmatrix}\boxed{2} & 1 & 1 & 1 & 0 & 0\\ 1 & 2 & 1 & 0 & 1 & 0\\ 1 & 1 & 2 & 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2 & 1 & 1 & 1 & 0 & 0\\ 0 & \boxed{3} & 1 & -1 & 2 & 0\\ 0 & 1 & 3 & -1 & 0 & 2\end{pmatrix}\begin{matrix}\\ R_2\to 2R_2-R_1 \quad \{(2\ 4\ 2\ 0\ 2\ 0)-(2\ 1\ 1\ 1\ 0\ 0)\}\\ R_3\to 2R_3-R_1 \quad \{(2\ 2\ 4\ 0\ 0\ 2)-(2\ 1\ 1\ 1\ 0\ 0)\}\end{matrix}$$

$$=\begin{pmatrix}6 & 0 & 2 & 4 & -2 & 0\\ 0 & 3 & 1 & -1 & 2 & 0\\ 0 & 0 & \boxed{8} & -2 & -2 & 6\end{pmatrix}\begin{matrix}R_1\to 3R_1-R_2 \quad \{(6\ 3\ 3\ 3\ 0\ 0)-(0\ 3\ 1\ -1\ 2\ 0)\}\\ \\ R_3\to 3R_3-R_2 \quad \{(0\ 3\ 9\ -3\ 0\ 6)-(0\ 3\ 1\ -1\ 2\ 0)\}\end{matrix}$$

$$=\begin{pmatrix} 24 & 0 & 0 & 18 & -6 & -6 \\ 0 & 24 & 0 & -6 & 18 & -6 \\ 0 & 0 & 8 & -2 & -2 & 6 \end{pmatrix} \begin{matrix} R_1 \to 4R_1 - R_3 \ \{(24\ \ 0\ \ 8\ \ 16\ \ -8\ \ 0) - (0\ \ 0\ \ 8\ \ -2\ \ -2\ \ 6)\} \\ R_2 \to 8R_2 - R_3 \ \{(0\ \ 24\ 8\ -8\ 16\ 0) + (0\ \ 0\ 8\ -2\ -2\ 6)\} \\ \ \end{matrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & \frac{18}{24} & \frac{-6}{24} & \frac{-6}{24} \\ 0 & 1 & 0 & \frac{-6}{24} & \frac{18}{24} & \frac{-6}{24} \\ 0 & 0 & 1 & \frac{-2}{8} & \frac{-2}{8} & \frac{6}{8} \end{pmatrix} \begin{matrix} R_1 \to \frac{R_1}{24} \\ R_2 \to \frac{R_2}{24} \\ R_3 \to \frac{R_3}{8} \end{matrix}$$

$$\text{Hence} \quad A^{-1} = \begin{pmatrix} \frac{18}{24} & \frac{-6}{24} & \frac{-6}{24} \\ \frac{-6}{24} & \frac{18}{24} & \frac{-6}{24} \\ \frac{-2}{8} & \frac{-2}{8} & \frac{6}{8} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Solution of Simultaneous Linear Algebraic Equations

Indirect Methods (Iterative Method)

1. Gauss Jacobi Method   2. Gauss Seidal Method

- The iteration will converge if the absolute values of the leading diagonal values of the coefficient matrix A of the system $AX = B$ are greater than the sum of absolute values of the other coefficients of that row. This condition is sufficient but not necessary.
- Round off errors is very small in iterative methods
- Iteration method is a self correction method. That is, any error made in computation, is corrected in the subsequent iteration
- The method will work for special system of equations only, as the convergence is not assured

Gauss Jacobi Method (To solve the system of simultaneous equations)
Working Rule:

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- Consider a set of simultaneous equations in 3 variables, say,

  $a_1x+b_1y+c_1z=d_1;\ a_2x+b_2y+c_2z=d_2;\ a_3x+b_3y+c_3z=d_3$

- Re-arrange the equations so that the coefficient matrix is diagonally dominant. $\begin{matrix}|a_1|\ge|b_1|+|c_1|\\ |b_2|\ge|a_2|+|c_2|\\ |c_3|\ge|a_3|+|b_3|\end{matrix}$.

  (At least one condition must be strictly >)

- Rewrite the equations as follows:

$$\begin{aligned} x&=\frac{1}{a_1}\left(d_1-b_1y-c_1z\right)\\ y&=\frac{1}{b_2}\left(d_2-a_2x-c_2z\right) \quad ....(1)\\ z&=\frac{1}{c_3}\left(d_3-a_3x-b_3y\right)\end{aligned}$$

- Put $x=0,\ y=0,\ z=0$ in the RHS of (1). We get the first approximated values $x_1,\ y_1,\ z_1$
- Then put $x=x_1,\ y=y_1,\ z=z_1$ in the RHS of (1). We get the second approximated values $x_2,\ y_2,\ z_2$. i.e. to get $(r+1)th$ iterative values, we use $(r)th$ iterative values.
- Continue this process till the required accuracy of root is obtained.

1. Solve the linear system $10x+y-z=11.19;\ x+10y+z=28.08;\ -x-y+10z=35.61$**, by Gauss Jacobi iteration method, correct to two decimal places.** **QC 20817 MA 8452 APR 2022**

The coefficient matrix of the given system is $A=\begin{pmatrix}10&1&-1\\1&10&1\\-1&-1&10\end{pmatrix}$ which is diagonally dominant.

Rewrite the equations as follows:

$$\left.\begin{aligned} x&=\frac{1}{10}\left(11.19-y+z\right)\\ y&=\frac{1}{10}\left(28.08-x-z\right)\\ z&=\frac{1}{10}\left(35.61-x-y\right)\end{aligned}\right| .....(1)$$

Iteration 1: Put $x=0,\ y=0,\ z=0$ in RHS of (1)

https://doi.org/10.5281/zenodo.15289885

$$x_1 = \frac{1}{10}(11.19 - 0 + 0) = 1.119$$
$$y_1 = \frac{1}{10}(28.08 - 0 - 0) = 2.808$$
$$z_1 = \frac{1}{10}(35.61 - 0 - 0) = 3.561$$

Iteration 2: Put $x = x_1,\ y = y_1,\ z = z_1$ in RHS of (1)

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = \frac{1}{10}(11.19 - 2.808 + 3.561) = 1.193$$
$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = \frac{1}{10}(28.08 - 1.119 - 3.561) = 2.34$$
$$z_2 = \frac{1}{10}(35.61 - x_1 - y_1) = \frac{1}{10}(35.61 - 1.119 + 2.808) = 3.72$$

Iteration 3: Put $x = x_2,\ y = y_2,\ z = z_2$ in RHS of (1)

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = \frac{1}{10}(11.19 - 2.34 + 3.729) = 1.2579$$
$$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = \frac{1}{10}(28.08 - 1.193 - 3.729) = 2.3158$$
$$z_3 = \frac{1}{10}(35.61 - x_2 - y_2) = \frac{1}{10}(35.61 - 1.193 + 2.34) = 3.6757$$

Iteration 4: Put $x = x_3,\ y = y_3,\ z = z_3$ in RHS of (1)

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = \frac{1}{10}(11.19 - 2.3158 + 3.6757) = 1.2549$$
$$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = \frac{1}{10}(28.08 - 1.2579 - 3.6757) = 2.3146$$
$$z_4 = \frac{1}{10}(35.61 - x_3 - y_3) = \frac{1}{10}(35.61 - 1.2579 + 2.3158) = 3.6667$$

Iteration 5: Put $x = x_4,\ y = y_4,\ z = z_4$ in RHS of (1)

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = \frac{1}{10}(11.19 - 2.3146 + 3.6667) = 1.2542$$
$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = \frac{1}{10}(28.08 - 1.2549 - 3.6667) = 2.3158$$
$$z_5 = \frac{1}{10}(35.61 - x_4 - y_4) = \frac{1}{10}(35.61 - 1.2549 + 2.3146) = 3.6669$$

Since $x_4 \approx x_5,\ y_4 \approx y_5,\ z_4 \approx z_5$, let the solution is $x = 1.2542,\ y = 2.3158,\ z = 3.6669$

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2. Solve, by Gauss-Jacobi method, the system of following equations correct to 3 decimal places.
$x+y+54z=110,\ 27x+6y-z=85,\ 6x+15y+2z=72.$ **QC 20753 MA 6452 NOV 2018**

The coefficient matrix of the given system is $A=\begin{pmatrix}27 & 6 & -1\\ 6 & 15 & 2\\ 1 & 1 & 54\end{pmatrix}$ which is diagonally dominant.

Rewrite the equations as follows:
$$\left.\begin{aligned} x&=\frac{1}{27}(85-6y+z)\\ y&=\frac{1}{15}(72-6x-2z)\\ z&=\frac{1}{54}(110-x-y)\end{aligned}\right|\text{.....(1)}$$

Iteration 1: Put $x=0,\ y=0,\ z=0$ in RHS of (1)

$$x_1=\frac{1}{27}(85-6(0)+(0))=3.1481$$
$$y_1=\frac{1}{15}(72-6(0)-2(0))=4.8$$
$$z_1=\frac{1}{54}(110-(0)-(0))=2.037$$

Iteration 2: Put $x=x_1,\ y=y_1,\ z=z_1$ in RHS of (1)

$$x_2=\frac{1}{27}(85-6y_1+z_1)=\frac{1}{27}(85-6(4.8)+2.037)=2.1569$$
$$y_2=\frac{1}{15}(72-6x_1-2z_1)=\frac{1}{15}(72-6(3.148)-2(2.037))=3.2692$$
$$z_2=\frac{1}{54}(110-x_1-y_1)=\frac{1}{54}(110-3.148-4.8)=1.8898$$

Iteration 3: Put $x=x_2,\ y=y_2,\ z=z_2$ in RHS of (1)

$$x_3=\frac{1}{27}(85-6y_2+z_2)=\frac{1}{27}(85-6(3.2692)+1.8898)=2.4916$$
$$y_3=\frac{1}{15}(72-6x_2-2z_2)=\frac{1}{15}(72-6(2.1569)-2(1.8898))=3.6853$$
$$z_3=\frac{1}{54}(110-x_2-y_2)=\frac{1}{54}(110-2.1569-3.2692)=1.9365$$

Iteration 4: Put $x=x_3,\ y=y_3,\ z=z_3$ in RHS of (1)

$$x_4 = \frac{1}{27}\left(85 - 6y_3 + z_3\right) = \frac{1}{27}\left(85 - 6(3.6853) + 1.9365\right) = 2.4009$$

$$y_4 = \frac{1}{15}\left(72 - 6x_3 - 2z_3\right) = \frac{1}{15}\left(72 - 6(2.4916) - 2(1.9365)\right) = 3.5452$$

$$z_4 = \frac{1}{54}\left(110 - x_3 - y_3\right) = \frac{1}{54}\left(110 - 2.4916 - 3.6853\right) = 1.9226$$

Iteration 5: Put $x = x_4,\ y = y_4,\ z = z_4$ in RHS of (1)

$$x_5 = \frac{1}{27}\left(85 - 6y_4 + z_4\right) = \frac{1}{27}\left(85 - 6(3.5452) + 1.9226\right) = 2.4315$$

$$y_5 = \frac{1}{15}\left(72 - 6x_4 - 2z_4\right) = \frac{1}{15}\left(72 - 6(2.4009) - 2(1.9226)\right) = 3.5833$$

$$z_5 = \frac{1}{54}\left(110 - x_4 - y_4\right) = \frac{1}{54}\left(110 - 2.4009 - 3.5452\right) = 1.9269$$

Iteration 6: Put $x = x_5,\ y = y_5,\ z = z_5$ in RHS of (1)

$$x_6 = \frac{1}{27}\left(85 - 6y_5 + z_5\right) = \frac{1}{27}\left(85 - 6(3.5833) + 1.9269\right) = 2.4232$$

$$y_6 = \frac{1}{15}\left(72 - 6x_5 - 2z_5\right) = \frac{1}{15}\left(72 - 6(2.4315) - 2(1.9269)\right) = 3.5704$$

$$z_6 = \frac{1}{54}\left(110 - x_5 - y_5\right) = \frac{1}{54}\left(110 - 2.4315 - 3.5833\right) = 1.9256$$

Since $x_5 \approx x_6,\ y_5 \approx y_6,\ z_5 \approx z_6$, let the solution is $x = 2.4232,\ y = 3.5704,\ z = 1.9256$

3. Solve the following equations using Jacobi's iteration method

$28x + 4y - z = 32,\quad x + 3y + 10z = 24,\quad 2x + 17y + 4z = 35$ **QC 80610 MA 6452 NOV 2016**

The coefficient matrix of the given system is $A = \begin{pmatrix} 28 & 64 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix}$ which is diagonally dominant.

Rewrite the equations as follows:

$$\left.\begin{aligned} x &= \frac{1}{28}\left(32 - 4y + z\right) \\ y &= \frac{1}{17}\left(35 - 2x - 4z\right) \\ z &= \frac{1}{10}\left(24 - x - 3y\right) \end{aligned}\right| \ldots\ldots(1)$$

Iteration 1: Put $x = 0,\ y = 0,\ z = 0$ in RHS of (1)

https://doi.org/10.5281/zenodo.15289885

$$x_1 = \frac{1}{28}\left(32 - 4(0) + (0)\right) = 1.143$$
$$y_1 = \frac{1}{17}\left(35 - 2(0) - 4(0)\right) = 2.058$$
$$z_1 = \frac{1}{10}\left(24 - (0) - 3(0)\right) = 2.4$$

Iteration 2: Put $x = x_1,\ y = y_1,\ z = z_1$ in RHS of (1)

$$x_2 = \frac{1}{28}\left(32 - 4y_1 + z_1\right) = \frac{1}{28}\left(32 - 4(2.058) + (2.4)\right) = 0.9345$$
$$y_2 = \frac{1}{17}\left(35 - 2x_1 - 4z_1\right) = \frac{1}{17}\left(35 - 2(1.413) - 4(2.4)\right) = 1.327$$
$$z_2 = \frac{1}{10}\left(24 - x_1 - 3y_1\right) = \frac{1}{10}\left(24 - (1.413) - 3(2.058)\right) = 1.641$$

Iteration 3: Put $x = x_2,\ y = y_2,\ z = z_2$ in RHS of (1)

$$x_3 = \frac{1}{28}\left(32 - 4y_2 + z_2\right) = \frac{1}{28}\left(32 - 4(1.327) + (1.641)\right) = 1.011$$
$$y_3 = \frac{1}{17}\left(35 - 2x_2 - 4z_2\right) = \frac{1}{17}\left(35 - 2(0.9345) - 4(1.641)\right) = 1.563$$
$$z_3 = \frac{1}{10}\left(24 - x_2 - 3y_2\right) = \frac{1}{10}\left(24 - (0.9345) - 3(1.327)\right) = 1.908$$

Iteration 4: Put $x = x_3,\ y = y_3,\ z = z_3$ in RHS of (1)

$$x_4 = \frac{1}{28}\left(32 - 4y_3 + z_3\right) = \frac{1}{28}\left(32 - 4(1.563) + (1.908)\right) = 0.9877$$
$$y_4 = \frac{1}{17}\left(35 - 2x_3 - 4z_3\right) = \frac{1}{17}\left(35 - 2(1.011) - 4(1.908)\right) = 1.491$$
$$z_4 = \frac{1}{10}\left(24 - x_3 - 3y_3\right) = \frac{1}{10}\left(24 - (1.011) - 3(1.563)\right) = 1.83$$

Iteration 5: Put $x = x_4,\ y = y_4,\ z = z_4$ in RHS of (1)

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$$x_5 = \frac{1}{28}(32 - 4y_4 + z_4) = \frac{1}{28}(32 - 4(1.491) + (1.83)) = 0.9952$$
$$y_5 = \frac{1}{17}(35 - 2x_4 - 4z_4) = \frac{1}{17}(35 - 2(0.9877) - 4(1.83)) = 1.512$$
$$z_5 = \frac{1}{10}(24 - x_4 - 3y_4) = \frac{1}{10}(24 - (0.9877) - 3(1.491)) = 1.853$$

Iteration 6: Put $x = x_5,\ y = y_5,\ z = z_5$ in RHS of (1)

$$x_6 = \frac{1}{28}(32 - 4y_5 + z_5) = \frac{1}{28}(32 - 4(1.512) + (1.853)) = 0.993$$
$$y_6 = \frac{1}{17}(35 - 2x_5 - 4z_5) = \frac{1}{17}(35 - 2(0.9952) - 4(1.853)) = 1.505$$
$$z_6 = \frac{1}{10}(24 - x_5 - 3y_5) = \frac{1}{10}(24 - (0.9952) - 3(1.512)) = 1.846$$

Since $x_5 \approx x_6,\ y_5 \approx y_6,\ z_5 \approx z_6$, let the solution is $x = 0.993,\ y = 1.505,\ z = 1.846$

Gauss Seidal Method

(To solve the system of simultaneous equations)

Working Rule:

- Consider a set of simultaneous equations in 3 variables, say,

  $a_1x + b_1y + c_1z = d_1;\ a_2x + b_2y + c_2z = d_2;\ a_3x + b_3y + c_3z = d_3$

- Re-arrange the equations so that the coefficient matrix is diagonally dominant. $\begin{matrix} |a_1| \geq |b_1| + |c_1| \\ |b_2| \geq |a_2| + |c_2| \\ |c_3| \geq |a_3| + |b_3| \end{matrix}$.

  (At least one condition must be strictly >)

- Rewrite the equations as follows:

https://doi.org/10.5281/zenodo.15289885

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \ldots.(1)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \ldots.(2)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \ldots.(3)$$

- Iteration-1: Put $y = 0,\ z = 0$ in the RHS of (1), we x_1
- Then put $x = x_1,\ z = 0$ in the RHS of (2), we get y_1.
- Then put $x = x_1,\ y = y_1$ in the RHS of (3), we get z_1.
- Similarly for every iteration, substitute the latest obtained values of $x,\ y,\ z$ to get the subsequent values.
- Continue this process till the required accuracy of root is obtained.

1. State the rate of convergence of Gauss Jacobi method and Gauss Seidal method?
QC 50785 MA 6459 DEC 2017

Rate of convergence of Gauss Jacobi method is 1 and Gauss Seidal method is 1.7

2. Which of the iteration method for solving linear system of equation converges faster? Why? **QC 80612 MA 6459 DEC 2016**

Gauss Seidal method is faster, because its rate of convergence is roughly 2 times than that of Gauss Jacobi method.

Since the most recent approximations of the variables are used, the convergence in the Gauss Seidal method is fast than the Gauss Jacobi method.

3. Write a sufficient condition for Gauss Seidal method to convergence. **QC 41313 MA 6452 MAY 18**

The coefficient matrix of the given system must be diagonally dominant.

4. Distinguish between Gauss elimination and Gauss-Seidal methods. **QC 50782 MA 6452 NOV 2017**

Gauss Elimination	Gauss Seidal
Direct method	Indirect method
Exact value can be obtained	Only approximate value can be obtained
No restriction about coefficient matrix	The coefficient matrix should be diagonally dominant.
Performance is affected by round off error.	Error will be controlled by increasing number of iterations

Round off errors get accumulated in every calculation and will be large	Round off errors will occur in the final iteration and will be small
Solution is obtained in finite number of steps	Number of iterations depends on the accuracy of the root is desired
No initial roots are required	Initial roots are required

5 Why is Gauss Seidal method is better than Gauss Jordan method. **QC 41316 MA 6459 MAY 2018**

Iterative method is suitable for large number of equations
Iterative method is self corrective method. Hence round off error will be corrected in the subsequent iterations.

6. Solve the following system of equations using Gauss-Seidal iterative method
$8x - y + z = 18,\ \ 2x + 5y - 2z = 3,\ \ x + y - 3z = -6$ **QC 80610 MA 6452 NOV 2016**

Rearrange the equations as follows $8x - y + z = 18,\ \ 2x + 5y - 2z = 3,\ \ x + y - 3z = -6$, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 8 & -1 & 1 \\ 2 & 5 & -2 \\ 1 & 1 & -3 \end{pmatrix}$ diagonally dominant.

Rewrite the equations as follows:

$$x = \frac{1}{8}(18 + y - z) \ldots\ldots.(1)$$
$$y = \frac{1}{5}(3 - 2x + 2z) \ldots.(2)$$
$$z = \frac{1}{3}(6 + x + y) \ldots\ldots(3)$$

Iteration 1:

Put $y = 0,\ z = 0$ *in the RHS of* (1)

$$x_1 = \frac{1}{8}(18 + (0) - (0)) = 2.25$$

Put $x = x_1,\ z = 0$ *in the RHS of* (2)

$$y_1 = \frac{1}{5}(3 - 2x_1 + 2(0)) = \frac{1}{5}(3 - 2(2.25) + 2(0)) = -0.3$$

Put $x = x_1,\ y = y_1$ *in the RHS of* (3)

$$z_1 = \frac{1}{3}(6 + x_1 + y_1) = \frac{1}{3}(6 + (2.25) + (-0.3)) = 2.65$$

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Iteration 2:

$Put\ y = y_1,\ z = z_1\ in\ the\ RHS\ of\ (1)$

$$x_2 = \frac{1}{8}(18 + y_1 - z_1) = \frac{1}{8}(18 - 0.3 - 2.65) = 1.881$$

$Put\ x = x_2,\ z = z_1\ in\ the\ RHS\ of\ (2)$

$$y_2 = \frac{1}{5}(3 - 2x_2 + 2z_1) = \frac{1}{5}(3 - 2(1.881) + 2(2.65)) = 0.9076$$

$Put\ x = x_2,\ y = y_2\ in\ the\ RHS\ of\ (3)$

$$z_2 = \frac{1}{3}(6 + x_2 + y_2) = \frac{1}{3}(6 + (1.881) + (0.9076)) = 2.9295$$

Iteration 3:

$Put\ y = y_2,\ z = z_2\ in\ the\ RHS\ of\ (1)$

$$x_3 = \frac{1}{8}(18 + y_2 - z_2) = \frac{1}{8}(18 + 0.9076 - 2.9295) = 1.9973$$

$Put\ x = x_3,\ z = z_2\ in\ the\ RHS\ of\ (2)$

$$y_3 = \frac{1}{5}(3 - 2x_3 + 2z_2) = \frac{1}{5}(3 - 2(1.9973) + 2(2.9295)) = 0.9728$$

$Put\ x = x_3,\ y = y_3\ in\ the\ RHS\ of\ (3)$

$$z_3 = \frac{1}{3}(6 + x_3 + y_3) = \frac{1}{3}(6 + (1.9973) + (0.9728)) = 2.99$$

Iteration 4:

$Put\ y = y_3,\ z = z_3\ in\ the\ RHS\ of\ (1)$

$$x_4 = \frac{1}{8}(18 + y_3 - z_3) = \frac{1}{8}(18 + 0.9728 - 2.99) = 1.9978$$

$Put\ x = x_4,\ z = z_3\ in\ the\ RHS\ of\ (2)$

$$y_4 = \frac{1}{5}(3 - 2x_4 + 2z_3) = \frac{1}{5}(3 - 2(1.9978) + 2(2.99)) = 0.9968$$

$Put\ x = x_4,\ y = y_4\ in\ the\ RHS\ of\ (3)$

$$z_4 = \frac{1}{3}(6 + x_4 + y_4) = \frac{1}{3}(6 + (1.9978) + (0.9968)) = 2.9982$$

Iteration 5:

https://doi.org/10.5281/zenodo.15289885

Put $y = y_4,\ z = z_4$ *in the RHS of* (1)

$$x_5 = \frac{1}{8}(18 + y_4 - z_4) = \frac{1}{8}(18 + 0.9968 - 2.9982) = 1.9998$$

Put $x = x_5,\ z = z_4$ *in the RHS of* (2)

$$y_5 = \frac{1}{5}(3 - 2x_5 + 2z_4) = \frac{1}{5}(3 - 2(1.9998) + 2(2.9982)) = 0.9994$$

Put $x = x_5,\ y = y_5$ *in the RHS of* (3)

$$z_5 = \frac{1}{3}(6 + x_5 + y_5) = \frac{1}{3}(6 + (1.9998) + (0.9994)) = 2.9997$$

Since $x_4 \approx x_5,\ y_4 \approx y_5,\ z_4 \approx z_5$, the solution is $x = 1.99978,\ y = 0.9994,\ z = 2.9997$

7. Solve the following system of equations by Gauss- Seidal method correct to 3 decimal places. $x + y + 54z = 110;\quad 27x + 6y - z = 85;\quad 6x + 15y + 2z = 72$ **QC 41313 MA 6452 MAY 2018**

Rearrange the equations as follows $27x + 6y - z = 85;\quad 6x + 15y + 2z = 72;\quad x + y + 54z = 110$, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{pmatrix}$ diagonally dominant.

Rewrite the equations as follows:

$$x = \frac{1}{27}(85 - 6y + z)\ldots\ldots.(1)$$

$$y = \frac{1}{15}(72 - 6x - 2z)\ldots.(2)$$

$$z = \frac{1}{54}(110 - x - y)\ldots\ldots(3)$$

Iteration 1:

Put $y = 0,\ z = 0$ *in the RHS of* (1)

$$x_1 = \frac{1}{27}(85 - 6(0) + (0)) = 3.1481$$

Put $x = x_1,\ z = 0$ *in the RHS of* (2)

$$y_1 = \frac{1}{15}(72 - 6x_1 - 2(0)) = 3.541$$

Put $x = x_1,\ y = y_1$ *in the RHS of* (3)

$$z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.9131$$

Iteration 2:

Put $y = y_1,\ z = z_1$ *in the RHS of* (1)

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.4321$$

Put $x = x_2,\ z = z_1$ *in the RHS of* (2)

$$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.572$$

Put $x = x_2,\ y = y_2$ *in the RHS of* (3)

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.9258$$

Iteration 3:

Put $y = y_2,\ z = z_2$ *in the RHS of* (1)

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.4256$$

Put $x = x_3,\ z = z_2$ *in the RHS of* (2)

$$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.5729$$

Put $x = x_3,\ y = y_3$ *in the RHS of* (3)

$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.9259$$

Iteration 4:

Iteration 5:

https://doi.org/10.5281/zenodo.15289885

Put $y = y_3,\ z = z_3$ in the RHS of (1)

$$x_4 = \frac{1}{27}(85 - 6y_3 + z_3) = 2.4255$$

Put $x = x_4,\ z = z_3$ in the RHS of (2)

$$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.3573$$

Put $x = x_4,\ y = y_4$ in the RHS of (3)

$$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.9299$$

Put $y = y_4,\ z = z_4$ in the RHS of (1)

$$x_5 = \frac{1}{27}(85 - 6y_4 + z_4) = 2.4735$$

Put $x = x_5,\ z = z_4$ in the RHS of (2)

$$y_5 = \frac{1}{15}(72 - 6x_5 - 2z_4) = 3.3553$$

Put $x = x_5,\ y = y_5$ in the RHS of (3)

$$z_5 = \frac{1}{54}(110 - x_5 - y_5) = 1.929$$

Since $x_4 \approx x_5,\ y_4 \approx y_5,\ z_4 \approx z_5$, let the solution is $x = 2.4735,\ y = 3.3553,\ z = 1.929$

8. Use Gauss-seidal iterative method to obtain the solution of the equations: $28x + 4y - z = 32;\ x + 3y + 10z = 24;\ 2x + 17y + 4z = 35$**, correct to 4 decimal accuracy.** **QC 60045 MA 3251 APR 2022**

Rearrange the equations, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 28 & 4 & 1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix}$ diagonally dominant.

Rewrite the equations as follows:

$$x = \frac{1}{28}(32 - 4y + z) \ldots\ldots.(1)$$

$$y = \frac{1}{17}(35 - 2x - 4z) \ldots.(2)$$

$$z = \frac{1}{10}(24 - x - 3y) \ldots\ldots(3)$$

Iteration 1:

$$Put\ \ y=0,\ z=0\ in\ the\ RHS\ of\ \ (1)$$

$$x_1=\frac{1}{28}(32-4y+z)=\frac{1}{28}(32-0+0)=1.1428$$

$$Put\ \ x=x_1,\ z=0\ in\ the\ RHS\ of\ \ (2)$$

$$y_1=\frac{1}{17}(35-2x_1-4z)=\frac{1}{17}(35-2.2857-0)=1.9243$$

$$Put\ \ x=x_1,\ y=y_1\ in\ the\ RHS\ of\ \ (3)$$

$$z_1=\frac{1}{10}(24-x_1-y_1)=\frac{1}{10}(24-1.1428+1.9243)=2.4781$$

Iteration 2:

$$Put\ \ y=y_1,\ z=z_1\ in\ the\ RHS\ of\ \ (1)$$

$$x_2=\frac{1}{28}(32-4y_1+z_1)=\frac{1}{28}(32-7.6972+2.4781)=0.9564$$

$$Put\ \ x=x_2,\ z=z_1\ in\ the\ RHS\ of\ \ (2)$$

$$y_2=\frac{1}{17}(35-2x_2-4z_1)=\frac{1}{17}(35-1.9129-9.9124)=1.3632$$

$$Put\ \ x=x_2,\ y=y_2\ in\ the\ RHS\ of\ \ (3)$$

$$z_2=\frac{1}{10}(24-x_2-y_2)=\frac{1}{10}(24-0.9564+1.3632)=2.4406$$

Iteration 3:

$$Put\ \ y=y_2,\ z=z_2\ in\ the\ RHS\ of\ \ (1)$$

$$x_3=\frac{1}{28}(32-4y_2+z_2)=\frac{1}{28}(32-5.4528+2.4406)=1.0352$$

$$Put\ \ x=x_3,\ z=z_2\ in\ the\ RHS\ of\ \ (2)$$

$$y_3=\frac{1}{17}(35-2x_3-4z_2)=\frac{1}{17}(35-2.0705-9.7624)=1.3627$$

$$Put\ \ x=x_3,\ y=y_3\ in\ the\ RHS\ of\ \ (3)$$

$$z_3=\frac{1}{10}(24-x_3-y_3)=\frac{1}{10}(24-1.0352+1.3627)=2.4327$$

Iteration 4:

https://doi.org/10.5281/zenodo.15289885

Put $y = y_3,\ z = z_3$ *in the RHS of* (1)

$$x_4 = \frac{1}{28}\left(32 - 4y_3 + z_3\right) = \frac{1}{28}\left(32 - 5.4508 + 2.4327\right) = 1.035$$

Put $x = x_4,\ z = z_3$ *in the RHS of* (2)

$$y_4 = \frac{1}{17}\left(35 - 2x_4 - 4z_3\right) = \frac{1}{17}\left(35 - 2.07 - 9.7308\right) = 1.3646$$

Put $x = x_4,\ y = y_4$ *in the RHS of* (3)

$$z_4 = \frac{1}{10}\left(24 - x_4 - y_4\right) = \frac{1}{10}\left(24 - 1.035 + 1.3646\right) = 2.4329$$

Iteration 5:

Put $y = y_4,\ z = z_4$ *in the RHS of* (1)

$$x_5 = \frac{1}{28}\left(32 - 4y_4 + z_4\right) = \frac{1}{28}\left(32 - 5.4584 + 2.4329\right) = 1.0348$$

Put $x = x_5,\ z = z_4$ *in the RHS of* (2)

$$y_5 = \frac{1}{17}\left(35 - 2x_5 - 4z_4\right) = \frac{1}{17}\left(35 - 2.0696 - 9.7316\right) = 1.3646$$

Put $x = x_5,\ y = y_5$ *in the RHS of* (3)

$$z_5 = \frac{1}{10}\left(24 - x_5 - y_5\right) = \frac{1}{10}\left(24 - 1.0348 + 1.3646\right) = 2.4329$$

Since $x_4 \approx x_5,\ y_4 \approx y_5,\ z_4 \approx z_5$, let the solution is $x = 1.0348,\ y = 1.3646,\ z = 2.4329$

9. Using Gauss-Seidal method solve the equations correct to 2 decimal places:

$10x_1 - 2x_2 - x_3 - x_4 = 3,\ -2x_1 + 10x_2 - x_3 - x_4 = 15,\ -x_1 - x_2 + 10x_3 - 2x_4 = 27,\ -x_1 - x_2 - 2x_3 + 10x_4 = -9$

QC 72071 MA 6452 MAY 2017

Rearrange the equations, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 10 & -2 & -1 & -1 \\ -2 & 10 & -1 & -1 \\ -1 & -1 & 10 & -2 \\ -1 & -1 & -2 & 10 \end{pmatrix}$

diagonally dominant.

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Rewrite the equations as follows:

$$x_1 = \frac{1}{10}\left(3 + 2x_2 + x_3 + x_4\right) \ldots\ldots.(1)$$

$$x_2 = \frac{1}{10}\left(15 + 2x_1 + x_3 + x_4\right) \ldots.(2)$$

$$x_3 = \frac{1}{10}\left(27 + x_1 + x_2 + 2x_4\right) \ldots\ldots(3)$$

$$x_4 = \frac{1}{10}\left(-9 + x_1 + x_2 + 2x_3\right) \ldots..(4)$$

Iteration 1:

Substitute $x_2 = 0,\ x_3 = 0,\ x_4 = 0$ *in the RHS of* (1)

$$x_1^1 = \frac{1}{10}\left(3 + 2(0) + (0) + (0)\right) = 0.3$$

Substitute $x_1 = x_1^1,\ x_3 = 0,\ x_4 = 0$ *in the RHS of* (2)

$$x_2^1 = \frac{1}{10}\left(15 + 2(0.3) + (0) + (0)\right) = 1.56$$

Substitute $x_1 = x_1^1,\ x_2 = x_2^1,\ x_4 = 0$ *in the RHS of* (3)

$$x_3^1 = \frac{1}{10}\left(27 + 0.3 + 1.56 + 2(0)\right) = 2.886$$

Substitute $x_1 = x_1^1,\ x_2 = x_2^1,\ x_3 = x_3^1$ *in the RHS of* (4)

$$x_4^1 = \frac{1}{10}\left(-9 + 0.3 + 1.56 + 2(2.866)\right) = -0.1408$$

Iteration 2:

Substitute $x_2 = x_2^1,\ x_3 = x_3^1,\ x_4 = x_4^1$ *in the RHS of* (1)

$$x_1^2 = \frac{1}{10}\left(3 + 2\left(x_2^1\right) + \left(x_3^1\right) + \left(x_4^1\right)\right) = \frac{1}{10}\left(3 + 2\left(1.56\right) + \left(2.866\right) + \left(-0.1408\right)\right) = 0.8845$$

Substitute $x_1 = x_1^2,\ x_3 = x_3^1,\ x_4 = x_4^1$ *in the RHS of* (2)

$$x_2^2 = \frac{1}{10}\left(15 + 2\left(x_1^2\right) + \left(x_3^1\right) + \left(x_4^1\right)\right) = \frac{1}{10}\left(15 + 2\left(0.8845\right) + \left(2.886\right) + \left(-0.1408\right)\right) = 1.951$$

Substitute $x_1 = x_1^2,\ x_2 = x_2^2,\ x_4 = x_4^1$ *in the RHS of* (3)

$$x_3^2 = \frac{1}{10}\left(27 + \left(x_1^2\right) + \left(x_2^2\right) + 2\left(x_4^1\right)\right) = \frac{1}{10}\left(27 + 0.8445 + 1.951 + 2\left(-0.1408\right)\right) = 2.951$$

Substitute $x_1 = x_1^2,\ x_2 = x_2^2,\ x_3 = x_3^2$ *in the RHS of* (4)

$$x_4^2 = \frac{1}{10}\left(-9 + \left(x_1^2\right) + \left(x_2^2\right) + 2\left(x_3^2\right)\right) = \frac{1}{10}\left(-9 + 0.8845 + 1.951 + 2\left(2.951\right)\right) = -0.0263$$

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Iteration 3:

$$Substitute\ x_2 = x_2^2,\ x_3 = x_3^2,\ x_4 = x_4^2\ in\ the\ RHS\ of\ (1)$$

$$x_1^3 = \frac{1}{10}\left(3+2\left(x_2^2\right)+\left(x_3^2\right)+\left(x_4^2\right)\right) = \frac{1}{10}\left(3+2(1.951)+(2.951)+(-0.0263)\right) = 0.9827$$

$$Substitute\ x_1 = x_1^3,\ x_3 = x_3^2,\ x_4 = x_4^2\ in\ the\ RHS\ of\ (2)$$

$$x_2^3 = \frac{1}{10}\left(15+2\left(x_1^3\right)+\left(x_3^2\right)+\left(x_4^2\right)\right) = \frac{1}{10}\left(15+2(0.9827)+(2.951)+(-0.0263)\right) = 1.989$$

$$Substitute\ x_1 = x_1^3,\ x_2 = x_2^3,\ x_4 = x_4^2\ in\ the\ RHS\ of\ (3)$$

$$x_3^3 = \frac{1}{10}\left(27+\left(x_1^3\right)+\left(x_2^3\right)+2\left(x_4^2\right)\right) = \frac{1}{10}\left(27+0.9827+1.989+2(-0.0263)\right) = 2.992$$

$$Substitute\ x_1 = x_1^3,\ x_2 = x_2^3,\ x_3 = x_3^3\ in\ the\ RHS\ of\ (4)$$

$$x_4^3 = \frac{1}{10}\left(-9+\left(x_1^3\right)+\left(x_2^3\right)+2\left(x_3^3\right)\right) = \frac{1}{10}\left(-9+0.9827+1.989+2(2.992)\right) = -0.0201$$

Iteration 4:

$$Substitute\ x_2 = x_2^3,\ x_3 = x_3^3,\ x_4 = x_4^3\ in\ the\ RHS\ of\ (1)$$

$$x_1^4 = \frac{1}{10}\left(3+2\left(x_2^3\right)+\left(x_3^3\right)+\left(x_4^3\right)\right) = \frac{1}{10}\left(3+2(1.989)+(2.992)+(-0.0201)\right) = 0.9949$$

$$Substitute\ x_1 = x_1^4,\ x_3 = x_3^3,\ x_4 = x_4^3\ in\ the\ RHS\ of\ (2)$$

$$x_2^4 = \frac{1}{10}\left(15+2\left(x_1^4\right)+\left(x_3^3\right)+\left(x_4^3\right)\right) = \frac{1}{10}\left(15+2(0.9949)+(2.992)+(-0.0201)\right) = 1.9962$$

$$Substitute\ x_1 = x_1^4,\ x_2 = x_2^4,\ x_4 = x_4^3\ in\ the\ RHS\ of\ (3)$$

$$x_3^4 = \frac{1}{10}\left(27+\left(x_1^4\right)+\left(x_2^4\right)+2\left(x_4^3\right)\right) = \frac{1}{10}\left(27+0.9949+1.9962+2(-0.0201)\right) = 2.9951$$

$$Substitute\ x_1 = x_1^4,\ x_2 = x_2^4,\ x_3 = x_3^4\ in\ the\ RHS\ of\ (4)$$

$$x_4^4 = \frac{1}{10}\left(-9+\left(x_1^4\right)+\left(x_2^4\right)+2\left(x_3^4\right)\right) = \frac{1}{10}\left(-9+0.9949+1.9962+2(2.9951)\right) = -0.0187$$

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Iteration 5:

$$Substitute\ x_2 = x_2^4,\ x_3 = x_3^4,\ x_4 = x_4^4\ in\ the\ RHS\ of\ (1)$$

$$x_1^5 = \frac{1}{10}\left(3+2\left(x_2^4\right)+\left(x_3^4\right)+\left(x_4^4\right)\right) = \frac{1}{10}\left(3+2\left(1.9962\right)+\left(2.9951\right)+\left(-0.0187\right)\right) = 0.9968$$

$$Substitute\ x_1 = x_1^5,\ x_3 = x_3^4,\ x_4 = x_4^4\ in\ the\ RHS\ of\ (2)$$

$$x_2^5 = \frac{1}{10}\left(15+2\left(x_1^5\right)+\left(x_3^4\right)+\left(x_4^4\right)\right) = \frac{1}{10}\left(15+2\left(0.9968\right)+\left(2.9951\right)+\left(-0.0187\right)\right) = 1.997$$

$$Substitute\ x_1 = x_1^5,\ x_2 = x_2^5,\ x_4 = x_4^4\ in\ the\ RHS\ of\ (3)$$

$$x_3^5 = \frac{1}{10}\left(27+\left(x_1^5\right)+\left(x_2^5\right)+2\left(x_4^4\right)\right) = \frac{1}{10}\left(27+0.9968+1.997+2\left(-0.0187\right)\right) = 2.9956$$

$$Substitute\ x_1 = x_1^5,\ x_2 = x_2^5,\ x_3 = x_3^5\ in\ the\ RHS\ of\ (4)$$

$$x_4^5 = \frac{1}{10}\left(-9+\left(x_1^5\right)+\left(x_2^5\right)+2\left(x_3^5\right)\right) = \frac{1}{10}\left(-9+0.9968+1.997+2\left(2.9956\right)\right) = -0.015$$

Since $x_1^4 \approx x_1^5,\ x_2^4 \approx x_2^5,\ x_3^4 \approx x_3^5,\ x_4^4 \approx x_4^5$ **, the solution is**

$x_1 \approx 0.9968,\ x_2 \approx 1.997,\ x_3 \approx 2.9956,\ x_4 \approx -0.15$

10. Solve the following equations by Gauss-Seidal method:

$x+y+9z=15;\quad x+17y-2z=48;\quad 30x-2y+3z=75$ **QC 57506 MA 6452 MAY 2016**

As the coefficient matrix is not diagonally dominant we rewrite the equations.

$$30x-2y+3z=75$$
$$x+17y-2z=48$$
$$x+y+9z=15$$

Since, the diagonal elements are dominant in the coefficient matrix, we write x , y , z as follows:

$$x=\frac{1}{30}\left[75+2y-3z\right].......(1)$$

$$y=\frac{1}{17}\left[48-x+2z\right].........(2)$$

$$z=\frac{1}{9}\left[15-x-y\right]...........(3)$$

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First Iteration

$$Put\ y = 0,\ z = 0\ in\ the\ RHS\ of\ (1)$$

$$x_1 = \frac{1}{30}\left(75 + 2y - 3z\right) = \frac{1}{30}\left(75 + 2(0) - 3(0)\right) = 2.5$$

$$Put\ x = x_1,\ z = 0\ in\ the\ RHS\ of\ (2)$$

$$y_1 = \frac{1}{17}\left(48 - x_1 + 2z\right) = \frac{1}{17}\left(48 - 2.5 + 2(0)\right) = 2.6765$$

$$Put\ x = x_1,\ y = y_1\ in\ the\ RHS\ of\ (3)$$

$$z_1 = \frac{1}{9}\left(15 - x_1 - y_1\right) = \frac{1}{9}\left(15 - 2.5 - 2.6765\right) = 1.0915$$

Second Iteration

Put $y = y_1,\ z = z_1$ *in the RHS of* (1)

$$x_2 = \frac{1}{30}\left[75 + 2y_1 - 3z_1\right] = \frac{1}{30}\left[75 + 2(2.6765) - 3(1.0915)\right] = 2.5693$$

Put $x = x_2,\ z = z_1$ *in the RHS of* (2)

$$y_2 = \frac{1}{17}\left[48 - x_2 + 2z_1\right] = \frac{1}{17}\left[48 - 2.5693 + 2(1.0915)\right] = 2.8008$$

Put $x = x_2,\ y = y_2$ *in the RHS of* (3)

$$z_2 = \frac{1}{9}\left[15 - x_2 - y_2\right] = \frac{1}{9}\left[15 - 2.5693 - 2.8008\right] = 1.0699$$

Third Iteration

Put $y = y_2,\ z = z_2$ *in the RHS of* (1)

$$x_3 = \frac{1}{30}\left[75 + 2y_2 - 3z_2\right] = \frac{1}{30}\left[75 + 2(2.5693) - 3(1.0699)\right] = 2.5800$$

Put $x = x_3,\ z = z_2$ *in the RHS of* (2)

$$y_3 = \frac{1}{17}\left[48 - x_3 + 2z_2\right] = \frac{1}{17}\left[48 - 2.5800 + 2(1.0699)\right] = 2.7976$$

Put $x = x_3,\ y = y_3$ *in the RHS of* (3)

$$z_3 = \frac{1}{9}\left[15 - x_3 - y_3\right] = \frac{1}{9}\left[15 - 2.5800 - 2.7976\right] = 1.0692$$

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Fourth Iteration

Put $y = y_3,\ z = z_3$ *in the RHS of* (1)

$$x_4 = \frac{1}{30}\left[75 + 2y_3 - 3z_3\right] = \frac{1}{30}\left[75 + 2(2.7976) - 3(1.0692)\right] = 2.5796$$

Put $x = x_4,\ z = z_3$ *in the RHS of* (2)

$$y_4 = \frac{1}{17}\left[48 - x_4 + 2z_3\right] = \frac{1}{17}\left[48 - 2.5796 + 2(1.0692)\right] = 2.7975$$

Put $x = x_4,\ y = y_4$ *in the RHS of* (3)

$$z_4 = \frac{1}{9}\left[15 - x_4 - y_4\right] = \frac{1}{9}\left[15 - 2.5796 - 2.7975\right] = 1.0692$$

Comparing the last two iterations, let the solution be $x = 2.5796, \quad y = 2.7975, \quad z = 1.0692$

11. Solve by Gauss Seidal method: $5x - 2y + z = -4; \quad x + 6y - 2z = -1; \quad 3x + y + 5z = 13$

QC 27331 MA 6452 NOV 2015

Rearrange the equations, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 5 & -2 & 1 \\ 1 & 6 & -2 \\ 3 & 1 & 5 \end{pmatrix}$ diagonally dominant.

Rewrite the equations as follows:

$$x = \frac{1}{5}\left(-4 + 2y - z\right) \ldots\ldots(1)$$

$$y = \frac{1}{6}\left(-1 - x + 2z\right) \ldots\ldots(2)$$

$$z = \frac{1}{5}\left(13 - 3x - 5y\right) \ldots\ldots(3)$$

Iteration 1:

Put $y = 0,\ z = 0$ *in the RHS of* (1)

$$x_1 = \frac{1}{5}\left(-4 + 2(0) - (0)\right) = -0.8$$

Put $x = x_1,\ z = 0$ *in the RHS of* (2)

$$y_1 = \frac{1}{6}\left(-1 - x_1 + 2(0)\right) = -0.0333$$

Put $x = x_1,\ y = y_1$ *in the RHS of* (3)

$$z_1 = \frac{1}{5}\left(13 - 3x_1 - 5y_1\right) = 3.1133$$

Iteration 2:

$$Put\ \ y = y_1,\ z = z_1\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_2 = \frac{1}{5}\left(-4 + 2y_1 - z_1\right) = -1.4358$$

$$Put\ \ x = x_2,\ z = z_1\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_2 = \frac{1}{6}\left(-1 - x_2 + 2z_1\right) = 1.1104$$

$$Put\ \ x = x_2,\ y = y_2\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_2 = \frac{1}{5}\left(13 - 3x_2 - 5y_2\right) = 2.2511$$

Iteration 3:

$$Put\ \ y = y_2,\ z = z_2\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_3 = \frac{1}{5}\left(-4 + 2y_2 - z_2\right) = \frac{1}{5}\left(-4 + 2(1.1104) - 2.2511\right) = -0.8061$$

$$Put\ \ x = x_3,\ z = z_2\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_3 = \frac{1}{6}\left(-1 - x_3 + 2z_2\right) = \frac{1}{6}\left(-1 + 0.8061 + 2(2.2511)\right) = 0.7181$$

$$Put\ \ x = x_3,\ y = y_3\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_3 = \frac{1}{5}\left(13 - 3x_3 - 5y_3\right) = \frac{1}{5}\left(13 + 3(0.8061) - 5(0.7181)\right) = 2.3656$$

Iteration 4:

$$Put\ \ y = y_3,\ z = z_3\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_4 = \frac{1}{5}\left(-4 + 2y_3 - z_3\right) = \frac{1}{5}\left(-4 + 2(0.7181) - 2.3656\right) = -0.9859$$

$$Put\ \ x = x_4,\ z = z_3\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_4 = \frac{1}{6}\left(-1 - x_4 + 2z_3\right) = \frac{1}{6}\left(-1 + 0.9859 + 2(2.3656)\right) = 0.7861$$

$$Put\ \ x = x_4,\ y = y_4\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_4 = \frac{1}{5}\left(13 - 3x_4 - 5y_4\right) = \frac{1}{5}\left(13 + 3(0.9859) - 5(0.7861)\right) = 2.4054$$

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Iteration 5:

Put $y = y_4,\ z = z_4$ *in the RHS of* (1)

$$x_5 = \frac{1}{5}\left(-4 + 2y_4 - z_4\right) = \frac{1}{5}\left(-4 + 2(0.7861) - 2.4054\right) = -0.9666$$

Put $x = x_5,\ z = z_4$ *in the RHS of* (2)

$$y_5 = \frac{1}{6}\left(-1 - x_5 + 2z_4\right) = \frac{1}{6}\left(-1 + 0.9666 + 2(2.4054)\right) = 0.7962$$

Put $x = x_5,\ y = y_5$ *in the RHS of* (3)

$$z_5 = \frac{1}{5}\left(13 - 3x_5 - 5y_5\right) = \frac{1}{5}\left(13 + 3(0.9666) - 5(0.7962)\right) = 2.3838$$

Iteration 6:

Put $y = y_5,\ z = z_5$ *in the RHS of* (1)

$$x_6 = \frac{1}{5}\left(-4 + 2y_5 - z_5\right) = \frac{1}{5}\left(-4 + 2(0.7962) - 2.3838\right) = -0.9583$$

Put $x = x_6,\ z = z_5$ *in the RHS of* (2)

$$y_6 = \frac{1}{6}\left(-1 - x_6 + 2z_5\right) = \frac{1}{6}\left(-1 + 0.9583 + 2(2.3838)\right) = 0.7877$$

Put $x = x_6,\ y = y_6$ *in the RHS of* (3)

$$z_6 = \frac{1}{5}\left(13 - 3x_6 - 5y_6\right) = \frac{1}{5}\left(13 + 3(0.9583) - 5(0.7877)\right) = 2.3873$$

Since $x_5 \approx x_6,\ y_5 \approx y_6,\ z_5 \approx z_6,$ the solution is $x = -0.9666,\ y = 0.7962,\ z = 2.3838$

12. Solve, by Gauss-Seidal method, the system of following equations correct to three decimal places $x + 3y + 10z = 24,\ 28x + 4y - z = 32,\ 2x + 17y + 4z = 35.$ **QC 20753 MA 6452 NOV 2018**

Rearrange the equations, so that the coefficient matrix of the given system is $A = \begin{pmatrix} 28 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{pmatrix}$ diagonally dominant.

Rewrite the equations as follows:

$$x = \frac{1}{28}\left(32 - 4y + z\right) \ldots\ldots.(1)$$

$$y = \frac{1}{17}\left(35 - 2x - 4z\right) \ldots.(2)$$

$$z = \frac{1}{10}\left(24 - x - 3y\right) \ldots\ldots(3)$$

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Iteration 1:

$$Put\ \ y=0,\ z=0\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_1=\frac{1}{28}\left(32-4(0)+(0)\right)=1.1428$$

$$Put\ \ x=x_1,\ z=0\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_1=\frac{1}{17}\left(35-2x_1-4(0)\right)=1.9244$$

$$Put\ \ x=x_1,\ y=y_1\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_1=\frac{1}{10}\left(24-x_1-3y_1\right)=1.7084$$

Iteration 2:

$$Put\ \ y=y_1,\ z=z_1\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_2=\frac{1}{28}\left(32-4y_1+z_1\right)=\frac{1}{28}\left(32-4(1.9244)+1.7084\right)=0.9289$$

$$Put\ \ x=x_2,\ z=z_1\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_2=\frac{1}{17}\left(35-2x_2-4z_1\right)=\frac{1}{17}\left(35-2(0.9289)-4(1.7084)\right)=1.5475$$

$$Put\ \ x=x_2,\ y=y_2\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_2=\frac{1}{10}\left(24-x_2-3y_2\right)=\frac{1}{10}\left(24-(0.9289)-3(1.5475)\right)=1.8428$$

Iteration 3:

$$Put\ \ y=y_2,\ z=z_2\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_3=\frac{1}{28}\left(32-4y_2+z_2\right)=\frac{1}{28}\left(32-4(1.5475)+1.8428\right)=0.9876$$

$$Put\ \ x=x_3,\ z=z_2\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_3=\frac{1}{17}\left(35-2x_3-4z_2\right)=\frac{1}{17}\left(35-2(0.9876)-4(1.8428)\right)=1.509$$

$$Put\ \ x=x_3,\ y=y_3\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_3=\frac{1}{10}\left(24-x_3-3y_3\right)=\frac{1}{10}\left(24-(0.9876)-3(1.509)\right)=1.8485$$

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Iteration 4:

$$Put\ \ y = y_3,\ z = z_3\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_4 = \frac{1}{28}(32 - 4y_3 + z_3) = \frac{1}{28}(32 - 4(1.509) + 1.8485) = 0.9933$$

$$Put\ \ x = x_4,\ z = z_3\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_4 = \frac{1}{17}(35 - 2x_4 - 4z_3) = \frac{1}{17}(35 - 2(0.9933) - 4(1.8485)) = 1.507$$

$$Put\ \ x = x_4,\ y = y_4\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_4 = \frac{1}{10}(24 - x_4 - 3y_4) = \frac{1}{10}(24 - (0.9933) - 3(1.507)) = 1.8485$$

Iteration 5:

$$Put\ \ y = y_4,\ z = z_4\ \ in\ the\ RHS\ of\ \ (1)$$

$$x_5 = \frac{1}{28}(32 - 4y_4 + z_4) = \frac{1}{28}(32 - 4(1.507) + 1.8485) = 0.9935$$

$$Put\ \ x = x_5,\ z = z_4\ \ in\ the\ RHS\ of\ \ (2)$$

$$y_5 = \frac{1}{17}(35 - 2x_5 - 4z_4) = \frac{1}{17}(35 - 2(0.9935) - 4(1.8485)) = 1.507$$

$$Put\ \ x = x_5,\ y = y_5\ \ in\ the\ RHS\ of\ \ (3)$$

$$z_5 = \frac{1}{10}(24 - x_5 - 3y_5) = \frac{1}{10}(24 - (0.9935) - 3(1.507)) = 1.8485$$

Since $x_4 \approx x_5,\ y_4 \approx y_5,\ z_4 \approx z_5,$ the solution is $x = 0.9935,\ y = 1.507,\ z = 1.8485$

Power Method

Power method is used to determine numerically largest eigenvalue and the corresponding eigenvector of a matrix A. Let A be a $n \times n$ matrix and let $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ be its distinct eigenvalues such that $|\lambda_1| > |\lambda_2| > |\lambda_3| > ...... > |\lambda_n|$. Also let $X_1, X_2, X_3, ...X_n$ be their independent eigen vectors. Then $AX_i = \lambda_i X_i,\ i = 1, 2, 3, ..., n$.

Define Power method. MA 6452 MAY 2019

Let X_0 be an arbitrary eigenvector and find $Y_{k+1} = AX_k$ and $X_{k+1} = \frac{Y_{k+1}}{m_{k+1}}$ where m_{k+1} is the numerically largest element of Y_{k+1}. Then $\lambda_1 = \underset{k \to \infty}{Lt} \frac{(Y_{k+1})_i}{(X_k)_i},\ i = 1, 2, ..., n$ and X_{k+1} is the required vector.

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Power method is used to determine numerically largest eigenvalue and the corresponding eigenvector of a matrix A. This method can be applied if the eigenvalues are real and the corresponding eigenvectors are linearly independent

What kind of an eigenvalue and eigenvector of a matrix would be obtained by Power method? MA 6452 NOV 2018

Results

- To find the smallest eigen value of A, obtain the numerically largest eigenvalue of A^{-1} and then take its reciprocal.
- Alternatively, obtain the largest eigenvalue λ_1 of A and then find $B = A - \lambda_1 I$ and find the dominant eigenvalue B. Then, the smallest eigenvalue of A is equal to the (dominant eigenvalue of B) $+\lambda_1$.

Power Method

(To find the numerically largest eigenvalue and its eigen vector)

Working Rule:

- Let A be a 3×3 matrix and let $X_0 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ *or* $\begin{pmatrix}0\\0\\1\end{pmatrix}$ *or* $\begin{pmatrix}0\\1\\0\end{pmatrix}$ *or* $\begin{pmatrix}1\\0\\0\end{pmatrix}$ be the initial eigenvector.
- Multiply AX_0 and the resultant product is $AX_0 = \begin{pmatrix}l\\m\\n\end{pmatrix}$ and let $l > m, n$.
- Then the highest term l can be taken out. i.e. $AX_0 = l\begin{pmatrix}1\\m/l\\n/l\end{pmatrix} = \lambda_1 X_1$
- Now multiply AX_1 and the resultant product is $AX_1 = \begin{pmatrix}p\\q\\r\end{pmatrix}$ and let $p > q, r$.
- Then the highest term p can be taken out. i.e. $AX_1 = p\begin{pmatrix}1\\q/p\\r/p\end{pmatrix} = \lambda_2 X_2$
- Continue this process for n times, say $AX_n = \lambda_n X_{n+1}$
- Then λ_n is the eigenvalue and X_{n+1} is the eigenvector

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Solved Problems

1. **From the following initial eigen vectors** $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ **which is the most suitable to find the largest eigen value of the matrix** $\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}$ **in one iteration.**

A.X = $\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix} . \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix} = 5\begin{pmatrix}1\\1\end{pmatrix} = \lambda X. \therefore \begin{pmatrix}1\\1\end{pmatrix}$ is the suitable initial eigen vector

2. **Using the Power method, find the largest eigenvalue and the corresponding eigenvector for the matrix** $A=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}$. **Let** $X_0=\begin{pmatrix}1\\0\end{pmatrix}$ **QC 41313 MA 6452 MAY 2018**

Let $X_0 = \begin{pmatrix}1\\0\end{pmatrix}$ be the initial eigen vector

$$AX_0=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}=3\begin{pmatrix}0.333\\1\end{pmatrix}=3\ X_1$$

$$AX_1=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0.333\\1\end{pmatrix}=\begin{pmatrix}4.333\\2.999\end{pmatrix}=4.333\begin{pmatrix}1\\0.6923\end{pmatrix}=4.999\ X_2$$

$$AX_2=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}1\\0.6923\end{pmatrix}=\begin{pmatrix}3.769\\4.384\end{pmatrix}=4.384\begin{pmatrix}0.8597\\1\end{pmatrix}=4.384\ X_3$$

$$AX_3=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0.8597\\1\end{pmatrix}=\begin{pmatrix}4.8597\\4.5791\end{pmatrix}=4.8597\begin{pmatrix}1\\0.9422\end{pmatrix}=4.8597X_4$$

$$AX_4=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}1\\0.9422\end{pmatrix}=\begin{pmatrix}4.7688\\4.8844\end{pmatrix}=4.8844\begin{pmatrix}0.9763\\1\end{pmatrix}=4.8844X_5$$

$$AX_5=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}0.9763\\1\end{pmatrix}=\begin{pmatrix}4.9763\\4.9289\end{pmatrix}=4.9763\begin{pmatrix}1\\0.9904\end{pmatrix}=4.9763X_6$$

$$AX_6=\begin{pmatrix}1 & 4\\3 & 2\end{pmatrix}\begin{pmatrix}1\\0.9904\end{pmatrix}=\begin{pmatrix}4.9618\\4.9808\end{pmatrix}=4.9808\begin{pmatrix}0.9962\\1\end{pmatrix}=4.9808X_7$$

Comparing the last 2 iterations, we conclude that the largest eigenvector is 4.9962≈5 and its eigenvector is $\begin{pmatrix}0.9962\\1\end{pmatrix} \square \begin{pmatrix}1\\1\end{pmatrix}$.

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3. **Find the dominant eigenvalue of** $A=\begin{pmatrix}2&3\\5&4\end{pmatrix}$ **by Power method, upto 1 decimal place accuracy. Start with** $X_0=\begin{pmatrix}1\\1\end{pmatrix}$ **QC 72074 MA 6459 MAY 2017**

Let $X_0 = \begin{pmatrix}1\\1\end{pmatrix}$ be the initial eigen vector

$$AX_0=\begin{pmatrix}2&3\\5&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\9\end{pmatrix} = 9\begin{pmatrix}0.555\\1\end{pmatrix} = 9X_1$$

$$AX_1=\begin{pmatrix}2&3\\5&4\end{pmatrix}\begin{pmatrix}0.555\\1\end{pmatrix}=\begin{pmatrix}4.111\\6.777\end{pmatrix}=6.777\begin{pmatrix}0.6066\\1\end{pmatrix}=6.777X_2$$

$$AX_2=\begin{pmatrix}2&3\\5&4\end{pmatrix}\begin{pmatrix}0.6066\\1\end{pmatrix}=\begin{pmatrix}4.213\\7.033\end{pmatrix}=7.033\begin{pmatrix}0.599\\1\end{pmatrix}=7.033X_3$$

$$AX_3=\begin{pmatrix}2&3\\5&4\end{pmatrix}\begin{pmatrix}0.599\\1\end{pmatrix}=\begin{pmatrix}4.198\\6.995\end{pmatrix}=6.995\begin{pmatrix}0.6\\1\end{pmatrix}=6.995X_4$$

$$AX_4=\begin{pmatrix}2&3\\5&4\end{pmatrix}\begin{pmatrix}0.6\\1\end{pmatrix}=\begin{pmatrix}4.2\\7\end{pmatrix}=7\begin{pmatrix}0.6\\1\end{pmatrix}=7X_5$$

Comparing the last 2 iterations, we conclude that the largest eigenvector is 7 and its eigenvector is $\begin{pmatrix}0.6\\1\end{pmatrix}$.

4. **Find the dominant eigenvalue of the matrix** $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ **and hence find the other eigenvalue also.** **QC 20817 MA 8452 APR 2022**

Let $X_0=\begin{pmatrix}1\\1\end{pmatrix}$ be the initial eigen vector

$$AX_0=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix} = 7\begin{pmatrix}0.4286\\1\end{pmatrix} = 7\ X_1$$

$$AX_1=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0.4286\\1\end{pmatrix} = \begin{pmatrix}2.4286\\5.2858\end{pmatrix} = 5.2858\begin{pmatrix}0.4595\\1\end{pmatrix} = 5.2858\ X_2$$

$$AX_2=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0.4595\\1\end{pmatrix} = \begin{pmatrix}2.4595\\5.378\end{pmatrix} = 5.378\begin{pmatrix}0.4573\\1\end{pmatrix} = 5.378\ X_3$$

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$$AX_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4573 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4573 \\ 5.372 \end{pmatrix} = 5.372\begin{pmatrix} 0.457 \\ 1 \end{pmatrix} = 5.372X_4$$

$$AX_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.457 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.457 \\ 5.371 \end{pmatrix} = 5.371\begin{pmatrix} 0.457 \\ 1 \end{pmatrix} = 5.371X_5$$

Since $X_4 \approx X_5$, λ = 5.4 is the largest eigen value of A.

Let $B = A - 5.4I = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 5.4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4.4 & 2 \\ 3 & -1.4 \end{pmatrix}$

Let X_0 = $X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the initial eigen vector

$$BX_0 = \begin{pmatrix} -4.4 & 2 \\ 3 & -1.4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2.4 \\ 2.6 \end{pmatrix} = -2.4\begin{pmatrix} 1 \\ -1.083 \end{pmatrix} = -2.4X_1$$

$$BX_1 = \begin{pmatrix} -4.4 & 2 \\ 3 & -1.4 \end{pmatrix}\begin{pmatrix} 1 \\ -1.083 \end{pmatrix} = \begin{pmatrix} -6.566 \\ 4.516 \end{pmatrix} = -6.566\begin{pmatrix} 1 \\ 0.688 \end{pmatrix} = -6.566X_2$$

$$BX_2 = \begin{pmatrix} -4.4 & 2 \\ 3 & -1.4 \end{pmatrix}\begin{pmatrix} 1 \\ -0.688 \end{pmatrix} = \begin{pmatrix} -5.775 \\ 3.9632 \end{pmatrix} = -5.775\begin{pmatrix} 1 \\ -0.686 \end{pmatrix} = -5.775X_3$$

$$BX_3 = \begin{pmatrix} -4.4 & 2 \\ 3 & -1.4 \end{pmatrix}\begin{pmatrix} 1 \\ -0.686 \end{pmatrix} = \begin{pmatrix} -5.772 \\ 3.9604 \end{pmatrix} = -5.772\begin{pmatrix} 1 \\ -0.686 \end{pmatrix} = -5.772X_4$$

Largest eigenvalue of B = –5.77 and hence smallest eigen value of A = 5.4 – 5.77 = –0.37

5. Find the dominant eigenvalue and its eigenvector of the matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ by Power method. Choose the initial vector as $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. QC 60045 MA 3251 APR 2022

Let the initial eigenvector be $X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$AX_0 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 10 \end{pmatrix} = 10\begin{pmatrix} -0.1 \\ 0.4 \\ 1 \end{pmatrix} = \lambda_1 X_1$$

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$$AX_1 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} -0.1 \\ 0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 4.5 \\ 10.26 \end{pmatrix} = 10.26 \begin{pmatrix} 0.009 \\ 0.4386 \\ 1 \end{pmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.009 \\ 0.4386 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4248 \\ 4.904 \\ 10.26 \end{pmatrix} = 10.26 \begin{pmatrix} 0.414 \\ 0.4779 \\ 1 \end{pmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.414 \\ 0.4779 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8477 \\ 6.1978 \\ 11.4976 \end{pmatrix} = 11.4976 \begin{pmatrix} 0.0737 \\ 0.539 \\ 1 \end{pmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.0737 \\ 0.539 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6907 \\ 5.2991 \\ 12.082 \end{pmatrix} = 12.082 \begin{pmatrix} 0.0572 \\ 0.4386 \\ 1 \end{pmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.0572 \\ 0.4386 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.373 \\ 5.0488 \\ 11.697 \end{pmatrix} = 11.697 \begin{pmatrix} 0.0318 \\ 0.4316 \\ 1 \end{pmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.0318 \\ 0.4316 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3266 \\ 4.9586 \\ 11.694 \end{pmatrix} = 11.694 \begin{pmatrix} 0.0279 \\ 0.424 \\ 1 \end{pmatrix} = \lambda_7 X_7$$

$$AX_7 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.0279 \\ 0.424 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.299 \\ 4.932 \\ 11.668 \end{pmatrix} = 11.668 \begin{pmatrix} 0.0256 \\ 0.4226 \\ 1 \end{pmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix} \begin{pmatrix} 0.0256 \\ 0.4226 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2934 \\ 4.922 \\ 11.664 \end{pmatrix} = 11.664 \begin{pmatrix} 0.0205 \\ 0.4219 \\ 1 \end{pmatrix} = \lambda_9 X_9$$

By comparing, the last two iterations, we conclude that the dominant eigenvalue is 11.664 and the corresponding eigenvector is $\begin{pmatrix} 0.0205 \\ 0.4219 \\ 1 \end{pmatrix}$.

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6. Find the numerically largest eigenvalue and the corresponding eigenvector of a matrix

$$A=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}$$

QC 50782 MA 6452 NOV 2017

Let the initial eigenvector be $X_0=\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$$AX_0=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}25\\1\\2\end{pmatrix}=25\begin{pmatrix}1\\0.04\\0.08\end{pmatrix}=\lambda_1 X_1$$

$$AX_1=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.04\\0.08\end{pmatrix}=\begin{pmatrix}25.2\\1.12\\1.68\end{pmatrix}=25.2\begin{pmatrix}1\\0.044\\0.066\end{pmatrix}=\lambda_2 X_2$$

$$AX_2=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.044\\0.066\end{pmatrix}=\begin{pmatrix}25.176\\1.132\\1.736\end{pmatrix}=25.176\begin{pmatrix}1\\0.0449\\0.0689\end{pmatrix}=\lambda_3 X_3$$

$$AX_3=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.0449\\0.0689\end{pmatrix}=\begin{pmatrix}26.1378\\1.1347\\1.7244\end{pmatrix}=26.1378\begin{pmatrix}1\\0.0434\\0.6597\end{pmatrix}=\lambda_4 X_4$$

$$AX_4=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.0434\\0.6597\end{pmatrix}=\begin{pmatrix}26.3628\\1.1302\\-0.6388\end{pmatrix}=26.3628\begin{pmatrix}1\\0.0428\\-0.0242\end{pmatrix}=\lambda_5 X_5$$

$$AX_5=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.0428\\-0.0242\end{pmatrix}=\begin{pmatrix}24.994\\1.1464\\2.0968\end{pmatrix}=24.994\begin{pmatrix}1\\0.0458\\0.0838\end{pmatrix}=\lambda_6 X_6$$

$$AX_6=\begin{pmatrix}25 & 1 & 2\\ 1 & 3 & 0\\ 2 & 0 & -4\end{pmatrix}\begin{pmatrix}1\\0.0458\\0.0838\end{pmatrix}=\begin{pmatrix}25.213\\1.1374\\1.6648\end{pmatrix}=25.213\begin{pmatrix}1\\0.0451\\0.066\end{pmatrix}=\lambda_7 X_7$$

$$AX_7 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0451 \\ 0.066 \end{pmatrix} = \begin{pmatrix} 25.177 \\ 1.1353 \\ 1.736 \end{pmatrix} = 25.177\begin{pmatrix} 1 \\ 0.045 \\ 0.0689 \end{pmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.045 \\ 0.0689 \end{pmatrix} = \begin{pmatrix} 25.1878 \\ 1.135 \\ 1.7244 \end{pmatrix} = 25.1878\begin{pmatrix} 1 \\ 0.045 \\ 0.0684 \end{pmatrix} = \lambda_9 X_9$$

By comparing, the last two iterations, we conclude that the dominant eigenvalue is 25.1878 and the corresponding eigenvector is $\begin{pmatrix} 1 \\ 0.045 \\ 0.0684 \end{pmatrix}$.

7. Determine by Power method the largest eigenvalue and the corresponding eigenvector of the matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$**.** **QC 72071 MA 6452 MAY 2017**

Let the initial eigenvector be $X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$AX_0 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 10 \end{pmatrix} = 10\begin{pmatrix} -0.1 \\ 0.4 \\ 1 \end{pmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} -0.1 \\ 0.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 4.5 \\ 10.26 \end{pmatrix} = 10.26\begin{pmatrix} 0.009 \\ 0.4386 \\ 1 \end{pmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.009 \\ 0.4386 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.4248 \\ 4.904 \\ 10.26 \end{pmatrix} = 10.26\begin{pmatrix} 0.414 \\ 0.4779 \\ 1 \end{pmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.414 \\ 0.4779 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8477 \\ 6.1978 \\ 11.4976 \end{pmatrix} = 11.4976\begin{pmatrix} 0.0737 \\ 0.539 \\ 1 \end{pmatrix} = \lambda_4 X_4$$

https://doi.org/10.5281/zenodo.15289885

$$AX_4 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0737 \\ 0.539 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6907 \\ 5.2991 \\ 12.082 \end{pmatrix} = 12.082\begin{pmatrix} 0.0572 \\ 0.4386 \\ 1 \end{pmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0572 \\ 0.4386 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.373 \\ 5.0488 \\ 11.697 \end{pmatrix} = 11.697\begin{pmatrix} 0.0318 \\ 0.4316 \\ 1 \end{pmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0318 \\ 0.4316 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3266 \\ 4.9586 \\ 11.694 \end{pmatrix} = 11.694\begin{pmatrix} 0.0279 \\ 0.424 \\ 1 \end{pmatrix} = \lambda_7 X_7$$

$$AX_7 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0279 \\ 0.424 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.299 \\ 4.932 \\ 11.668 \end{pmatrix} = 11.668\begin{pmatrix} 0.0256 \\ 0.4226 \\ 1 \end{pmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0256 \\ 0.4226 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2934 \\ 4.922 \\ 11.664 \end{pmatrix} = 11.664\begin{pmatrix} 0.0205 \\ 0.4219 \\ 1 \end{pmatrix} = \lambda_9 X_9$$

By comparing, the last two iterations, we conclude that the dominant eigenvalue is 11.664 and the corresponding eigenvector is $\begin{pmatrix} 0.0205 \\ 0.4219 \\ 1 \end{pmatrix}$.

8. Find the largest eigenvalue and eigenvector of the matrix $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by Power method.

QC 80610 MA 6452 NOV 2016

Let the initial eigenvector be $X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$AX_0 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0.333 \\ 0 \\ 1 \end{pmatrix} = \lambda_1 X_1$$

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$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0.333 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.333 \\ 0.333 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0.444 \\ 0.111 \\ 1 \end{pmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0.444 \\ 0.111 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.11 \\ 0.666 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 0.7033 \\ 0.222 \\ 1 \end{pmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0.7033 \\ 0.222 \\ 1 \end{pmatrix} = \begin{pmatrix} 3.0353 \\ 1.1473 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1.0117 \\ 0.3824 \\ 1 \end{pmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1.0117 \\ 0.3824 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.306 \\ 1.7765 \\ 3 \end{pmatrix} = 4.306\begin{pmatrix} 1 \\ 0.4125 \\ 0.6967 \end{pmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4125 \\ 0.6967 \end{pmatrix} = \begin{pmatrix} 4.1717 \\ 1.825 \\ 2.0901 \end{pmatrix} = 4.1717\begin{pmatrix} 1 \\ 0.4374 \\ 0.501 \end{pmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4374 \\ 0.501 \end{pmatrix} = \begin{pmatrix} 4.1254 \\ 1.8748 \\ 1.503 \end{pmatrix} = 4.1254\begin{pmatrix} 1 \\ 0.4544 \\ 0.3643 \end{pmatrix} = \lambda_7 X_7$$

$$AX_7 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4544 \\ 0.3643 \end{pmatrix} = \begin{pmatrix} 4.0907 \\ 1.9088 \\ 1.0929 \end{pmatrix} = 4.0907\begin{pmatrix} 1 \\ 0.4666 \\ 0.2672 \end{pmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4666 \\ 0.2672 \end{pmatrix} = \begin{pmatrix} 4.0668 \\ 1.9332 \\ 0.8016 \end{pmatrix} = 4.0668\begin{pmatrix} 1 \\ 0.4753 \\ 0.1971 \end{pmatrix} = \lambda_9 X_9$$

By comparing, the last two iterations, we conclude that the dominant eigenvalue is 4.0668 and the corresponding eigenvector is $\begin{pmatrix} 1 \\ 0.4753 \\ 0.1971 \end{pmatrix}$.

9. Find the largest eigenvalue and its corresponding eigenvector using Power method, for $A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}$

QC 53252 MA 6459 MAY 2019

Let the initial eigenvector be $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$AX_0 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 0.167 \\ 0.667 \\ 1 \end{pmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.167 \\ 0.667 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.166 \\ 2.336 \\ 8.003 \end{pmatrix} = 8.003\begin{pmatrix} 0.021 \\ 0.292 \\ 1 \end{pmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.021 \\ 0.292 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.145 \\ 0.252 \\ 6.002 \end{pmatrix} = 6.002\begin{pmatrix} 0.191 \\ 0.042 \\ 1 \end{pmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.191 \\ 0.042 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.065 \\ -0.068 \\ 6.272 \end{pmatrix} = 6.272\begin{pmatrix} 0.329 \\ -0.0911 \\ 1 \end{pmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.329 \\ -0.0911 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.362 \\ 0.272 \\ 6.941 \end{pmatrix} = 6.941\begin{pmatrix} 0.34 \\ 0.039 \\ 1 \end{pmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.34 \\ 0.039 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.223 \\ 0.516 \\ 7.157 \end{pmatrix} = 7.157\begin{pmatrix} 0.311 \\ 0.072 \\ 1 \end{pmatrix} = \lambda_6 X_6$$

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$$AX_6 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0.311 \\ 0.072 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.095 \\ 0.532 \\ 7.082 \end{pmatrix} = 7.082 \begin{pmatrix} 0.296 \\ 0.075 \\ 1 \end{pmatrix} = \lambda_7 X_7$$

$$AX_7 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0.296 \\ 0.075 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.071 \\ 0.484 \\ 7.001 \end{pmatrix} = 7.001 \begin{pmatrix} 0.296 \\ 0.069 \\ 1 \end{pmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0.296 \\ 0.069 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.089 \\ 0.46 \\ 6.983 \end{pmatrix} = 6.983 \begin{pmatrix} 0.296 \\ 0.066 \\ 1 \end{pmatrix} = \lambda_9 X_9$$

By comparing, the last two iterations, we conclude that the dominant eigenvalue is 6.983 and the corresponding eigenvector is $\begin{pmatrix} 0.296 \\ 0.066 \\ 1 \end{pmatrix}$.

Jacobi Method

<table>
<tr><td colspan="2">To find the eigenvalues & eigenvectors of a real symmetric matrix)</td></tr>
<tr><td>

Let A be a 2×2 matrix, say $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where $a_{21} = a_{12}$.

Let the rotation matrix be $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Find $\alpha = \dfrac{a_{11} - a_{22}}{2a_{12}}$, $\beta = \alpha \pm \sqrt{\alpha^2 + 1}$

Find

$\sin\theta = \dfrac{-1}{\sqrt{1+\beta^2}}$, $\cos\theta = \dfrac{\beta}{\sqrt{1+\beta^2}}$, $if\ \ \beta < 0$

$\sin\theta = \dfrac{1}{\sqrt{1+\beta^2}}$, $\cos\theta = \dfrac{\beta}{\sqrt{1+\beta^2}}$, $if\ \ \beta > 0$

The eigenvalues are given by

$b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta$

$b_{22} = a_{11} + a_{22} - b_{11}$

$(sum\, of\ eigenvalues = sum\, of\ diagonals)$.

The columns of R are eigenvectors

</td><td>

Another Method

Let A be a 2×2 matrix, say $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

where $a_{21} = a_{12}$.

Let the rotation matrix be $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Find $\theta = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2a_{12}}{a_{11} - a_{22}}\right)$

Find the diagonal matrix $B = R^T A R$

The diagonal elements of B are eigenvalues.

The columns of R are eigenvectors

</td></tr>
</table>

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Let A be a 3×3 symmetric matrix, say $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and let $a_{13}=a_{31}$ be the largest off diagonal element

Find θ such that $\cot 2\theta=\dfrac{a_{11}-a_{33}}{2a_{13}}$ *or* $\tan 2\theta=\dfrac{2a_{13}}{a_{11}-a_{33}}$

Find the first rotation matrix $S_1=\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$

Find the orthogonal matrix $B_1=S_1^T A S_1$. If B_1 is diagonal, the elements are the eigenvalues. Otherwise again choose the largest off diagonal element of B_1. Say $a_{12}=a_{21}$ be the largest off diagonal element

Again, find θ such that $\cot 2\theta=\dfrac{a_{11}-a_{22}}{2a_{12}}$

Then find the second rotation matrix $S_2=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find the matrix $B_2=S_2^T B_1 S_2$. If B_2 is diagonal, the elements are the eigenvalues. Otherwise continue the procedure once again.

The eigenvectors are given by the columns of the matrix $S=S_1\times S_2$

Note

- This method can be applied to find all the eigenvalues of a symmetric matrix
- A disadvantage of this method is that the element annihilated by a transformation, may not remain zero during the subsequent iterations.

1 Use Jacobi's method to find all the eigenvalues and eigenvectors of $A=\begin{pmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}$

Let $\alpha=\dfrac{a_{11}-a_{22}}{2a_{12}}=\dfrac{6-4}{2\sqrt{3}}=\dfrac{1}{\sqrt{3}}$ and $\beta=\alpha\pm\sqrt{\alpha^2+1}=\dfrac{1}{\sqrt{3}}\pm\sqrt{1+\dfrac{1}{3}}=\sqrt{3}$ *or* $-\dfrac{1}{\sqrt{3}}$

Take $\beta=\sqrt{3}$

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Now $\sin\theta = \frac{1}{\sqrt{1+\beta^2}} = \frac{1}{2}$ and $\cos\theta = \frac{\beta}{\sqrt{1+\beta^2}} = \frac{\sqrt{3}}{2}$

Hence $\theta = \frac{\pi}{6}$

$$b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta = 6\left(\frac{3}{4}\right) + \sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{4}\right) = 7$$

$b_{22} = a_{11} + a_{22} - b_{11} = 6 + 4 - 7 = 3$. Hence the eigenvalues are 7, 3

and the rotation matrix $R = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

The eigenvectors are the columns of R.

2. Find all the eigenvalues of $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ by Jacobi method. **QC 60045 MA 3251 APR 2022**

Let $\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(3)}{2-2} = \frac{6}{0} = \infty.$

Therefore $2\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ and rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Consider $R^T AR = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2+3 & 3+2 \\ -2+3 & -3+2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

https://doi.org/10.5281/zenodo.15289885

$$=\frac{1}{2}\begin{pmatrix} 5 & 5 \\ 1 & -1 \end{pmatrix}\times\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} 10 & 0 \\ 0 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence the eigenvalues are 5 and −1.

3. Find the eigenvalues and eigenvectors of the matrix $A=\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$

Here $a_{13}=a_{31}=2$ is the largest off diagonal element and hence $a_{11}=a_{33}=1$

Hence θ is given by $\cot 2\theta=\frac{a_{11}-a_{33}}{2a_{13}}=\frac{1-1}{2\times 4}=0$

Therefore $2\theta=\frac{\pi}{2},\ \theta=\frac{\pi}{4}$

Find the first rotation matrix $S_1=\begin{pmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

Find the matrix

$$B_1=S_1^T A S_1$$

$$=\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\times\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}\times\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{3}{\sqrt{2}} & 2 & \frac{3}{\sqrt{2}} \\ \sqrt{2} & 3 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}\times\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

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$$= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Again choose the largest off diagonal element of B_1. i.e. $a_{12} = a_{21} = 2$ be the largest off diagonal element

Again, find θ such that $\cot 2\theta = \frac{a_{11} - a_{22}}{2a_{12}} = \frac{3-3}{2\times 4} = 0$

Then find the second rotation matrix $S_2 = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find the matrix

$$B_2 = S_2^T B_1 S_2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Here B_2 is diagonal, hence the diagonal elements $5,\ 1,\ -1$ are the eigenvalues.

To find the eigenvectors:

Consider the matrix $S = S_1 \times S_2$

https://doi.org/10.5281/zenodo.15289885

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The columns of the matrix S are the corresponding eigenvectors.

4. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$**, using Jacobi method**

(QC 80221 MA 8491 May 2019)

Here $a_{23} = a_{32} = -1$ is the largest off diagonal element and hence $a_{22} = a_{33} = 3$

Hence θ is given by $\cot 2\theta = \frac{a_{22} - a_{33}}{2a_{23}} = \frac{3-3}{2\times(-1)} = 0$

Therefore $2\theta = \frac{\pi}{2},\ \theta = \frac{\pi}{4}$

Find the first rotation matrix $S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Find the matrix

$$B_1 = S_1^T A S_1$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -4 & 4 \end{pmatrix} \times \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Here B_1 is diagonal, hence the diagonal elements 1, 2, 4 are the eigenvalues.

To find the eigenvectors:

Consider the matrix $S = S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

The columns of the matrix S are the corresponding eigenvectors.

EXERCISE

Iterative Method

1. Find the positive root of $x^4 - x - 9 = 0$ using Iterative method.

2 Find the smallest positive root of the equation $e^{-x} = \sin x$ by iterative method.

Newton-Raphson Method

1. Find the positive root of $x^4 - x - 9 = 0$ using Newton method. (QC 27331 MA 6452 NOV 2015)

2 Find the smallest positive root of the equation $e^{-x} = \sin x$ by Newton's iterative method.

Gauss Elimination method

1 Solve the system of equations by Gauss-Elimination method. (QC 31528 MA 2266 NOV 2013)

https://doi.org/10.5281/zenodo.15289885

$x_1 + x_2 + x_3 + x_4 = 2;\ 2x_1 - x_2 + 2x_3 - x_4 = -5;\ 3x_1 + 2x_2 + 3x_3 + 4x_4 = 7;\ x_1 - 2x_2 - 3x_3 2x_4 = 5$

2 Solve by Gauss-Elimination method. (QC 31528 MA 2266 NOV 2013)

$3x + 4y + 5z = 18;\quad 2x - y + 8z = 13;\quad 5x - 2y + 7z = 20$

Gauss Jordan Method

1 Solve the system of equations by Gauss-Jordan method. (QC 21528 MA 2266 MAY 2013)

$x + y + z + w = 1;\quad 2x - y + 2z - w = -5;\quad 3x + 2y + 3z + 4w = 7;\quad x - 2y - 3z + 2w = 5$

1 Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ by Gauss Jordan method. (QCE3126 MA 2266 MAY 10)

2 Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ by Gauss-Jordan method. (QC 31528 MA 2266 NOV 2013)

Gauss Jacobi Method

1 Solve the following system of equations by Jacobi's method:

$8x - 3y + 2z = 20;\quad 6x + 3y + 12z = 35;\quad 4x + 11y - z = 33$

Gauss Seidal Method

1 Solve $5x - y + z = 10;\quad 2x + 4y = 12;\quad x + y + 5z = -1$ using Gauss Seidal method.

(QC E3126 MA 2266 MAY 2010)

2 Solve the following set of equations using Gauss Seidal iterative procedure

$-10x + 2y + 2z = 4;\quad x - 10y + 2z = 18;\quad x + y - 10z = 45$ (QC 51579 MA 2266 MAY 2014)

Power Method

1 Find the largest eigenvalue of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ by Power method. Also find its corresponding eigenvector. (QC 31528 MA 2266 NOV 2013)

2 Using Power method, find all the eigenvalues of $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$. (QC 21528 MA 2266 MAY 2013)

Jacobi Method

1. Find all the eigenvalues and eigenvectors of the following matrices by Jacobi's method

https://doi.org/10.5281/zenodo.15289885

(i) $A=\begin{pmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{pmatrix}$ (ii) $A=\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}$ (iii) $A=\begin{pmatrix}1 & 3\\ 3 & 4\end{pmatrix}$

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UNIT IV – INTERPOLATION, NUMERICAL DIFFERENTIATION AND INTEGRATION

Interpolation

Suppose we are given the numerical values of the polynomial $y = f(x)$ for a set of values of x as follows:

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	....	y_n

Interpolation is a technic of estimating the value of a function for any intermediate value of the independent variable. If the x value lies beyond the interval, it is known as extrapolation

Interpolation is the process of finding the values of a function y corresponding to any value of $x = x_i$ between x_0 and x_n.

What is meant by interpolation?
MA 6452 NOV 2017

Finite Differences

Consider the tabulated values of the function $y = f(x)$ for equally spaced values of $x = x_0,\ x_0 + h,$ $x_0 + 2h,\ ....,\ x_0 + nh$ as $y = y_0,\ y_1,\ y_2,\ ....,\ y_n$. To determine the polynomial or to determine interpolating values, the following differences are used.

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

Newton's Forward Difference Formula

While interpolating, if the value of x lies in the beginning of the table, use forward difference formula.

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ \ldots \text{ ,where } \quad u = \frac{x - x_0}{h}$$

Error of forward difference formula = f(x) – P_n(x) = $\dfrac{u(u-1)......(u-n)}{(n+1)!}\Delta^{n+1} f(c), \quad u = \dfrac{x - x_0}{h}$

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Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
		$y_3 - y_2 = \nabla y_3$		
x_3	y_3			

While interpolating, if the value of x lies at the end of the table, use backward difference formula.

$$y(x) = y_n + \frac{u}{1!}\Delta y_n + \frac{u(u+1)}{2!}\Delta^2 y_n + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_n + \ \ldots \text{ ,where } \quad u = \frac{x - x_n}{h}$$

Error in backward difference formula = f(x) – P_n(x) = $\frac{v(v+1)......(v+n)}{(n+1)!} h^{n+1} y^{n+1}(c), \quad v = \frac{x - x_n}{h}$

Note:

1. If the tabulated values of x and y satisfies an exact polynomial, then Newton’s forward and backward formulas gives the same answer

2. Newton forward/backward interpolation formula is used only for intervals of x which are equally spaced.

3. In the set of given tabulated values, if the value of x for which the value of y is to be evaluated lies at the beginning of the table, use Newton’s forward difference formula. If x lies at the end of the table, use Newton’s backward difference formula.

When to use Newton’s forward interpolation and when to use Newton’s backward interpolation formula? MA 6459 MAY 2018

Differences of a Polynomial

Consider a linear polynomial $y = f(x) = ax + b$. The forward differences of the polynomials are defined as follows:

$$\Delta f(x) = f(x+h) - f(x) = [a(x+h)+b] - [ax+b] = ah$$

$$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[ah] = ah - ah = 0$$

In general n^{th} difference of a n^{th} degree polynomial is $a.n!.h^n$ (constant) and higher difference are 0.

Similarly, we can define backward differences of the polynomials as follows: $\nabla f(x) = f(x) - f(x-h)$

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Solved Problems in Differences of Polynomials

1. **What is the form of the function tabulated at equally spaced intervals with sixth difference constant.**

   It is a polynomial of degree 6.

2. **What is the degree of interpolating polynomial if n values of x and y are given.**

   The degree of polynomial is $n-1$

3. **Evaluate $\Delta\left[x(x+1)(x+2)(x+3)\right]$** **QC 72071 MA 6452 MAY 2017**

   We know that $\Delta f(x) = f(x+h) - f(x)$ and let $h=1$

$$\Delta\left[x(x+1)(x+2)(x+3)\right]=\left[(x+1)(x+2)(x+3)(x+4)\right]-\left[x(x+1)(x+2)(x+3)\right]$$

$$=(x+1)(x+2)(x+3)\left[x+4-x\right]$$

$$=4(x+1)(x+2)(x+3)$$

4. **Find $\nabla^2(\sin x)$, where h is length of the interval.** **QC 80221 MA 8491 MAY 2019**

   We know that $\nabla f(x) = f(x) - f(x-h)$

   Consider

$$\nabla(\sin x) = \sin x - \sin(x-h)$$

$$\nabla^2(\sin x) = \nabla\left[\nabla(\sin x)\right]$$

$$= \nabla\left[\sin x - \sin(x-h)\right]$$

$$= \left[\sin x - \sin(x-h)\right] - \left[\sin(x-h) - \sin(x-h-h)\right]$$

$$= \sin x - 2\sin(x-h) + 2\sin(x-2h)$$

Solved Problems in Interpolation by Newton's Forward Difference

1. **Interpolate $y(12)$, if** **QC 57506 MA 6452 MAY 2016**

x :	**10**	**15**	**20**	**25**	**30**	**35**
y(x) :	**35**	**33**	**29**	**27**	**22**	**14**

Here x values are equally spaced. We have to find y at $x=12$. Also $x=12$ lies in the beginning of the table, hence we use Newton's forward difference formula.

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Forward difference Table:

x	y	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
10	35					
		−2				
15	33		−2			
		−4		4		
20	29		2		−9	
		−2		−5		14
25	27		−3		5	
		−5		0		
30	22		−3			
		−8				
35	14					

Forward formula

$$y(x) = y_0 + \frac{u}{1\ !}\Delta y_0 + \frac{u(u-1)}{2\ !}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\ !}\Delta^3 y_0 + \ldots \text{ ,where}$$

$$u = \frac{x-x_0}{h} = \frac{12-10}{5} = \frac{2}{5}$$

$$y(12) = 35 + \frac{\left(\frac{2}{5}\right)}{1!}(-2) + \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}-1\right)}{2!}(-2) + \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)}{3!}(4)$$

$$+ \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\left(\frac{2}{5}-3\right)}{4!}(-9) + \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\left(\frac{2}{5}-3\right)\left(\frac{2}{5}-4\right)}{5!}(14)$$

$$= 35 - 0.8 + 0.24 + 0.256 + 0.3744 + 0.419$$

$$= 35.4894$$

2. **Find the values of y at $x = 21$ from the following data.** **QC 20817 MA 8452 APR 2022**

x:	**20**	**23**	**26**	**29**
y:	**0.342**	**0.3907**	**0.4384**	**0.4848**

Here x values are equally spaced. Also $x = 21$ lies in the beginning of the table, hence we use Newton's forward difference formula.

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
23	0.3907		−0.001	
		0.0477		−0.0003
26	0.4384		−0.0013	
		0.0464		
29	0.4848			

Forward formula

$$y(x) = y_0 + \frac{u}{1\,!}\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \dots \text{,where}$$

$$u = \frac{x-x_0}{h} = \frac{21-20}{3} = \frac{1}{3}$$

$$y(21) = 0.342 + \frac{\left(\frac{1}{3}\right)}{1!}(0.0487) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}(-0.001) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-0.0003)$$

$$y(21) = 0.342 + 0.01623 + 0.00011 - 0.0000185$$

$$y(21) = 0.3585$$

3. Find the polynomial which takes the following values given $f(0) = -1,\ f(1) = 1$ and $f(2) = 4$ using the Newton's interpolating formula. **QC 80610 MA 6452 NOV 2016**

Forward difference Table:

x	y	Δy	$\Delta^2 y$
0	−1		
		2	
1	1		1
		3	
2	4		

Forward formula

$$y(x) = y_0 + \frac{u}{1\,!}\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \dots \text{, where}$$

$$u = \frac{x-x_0}{h} = \frac{x-0}{1} = x$$

https://doi.org/10.5281/zenodo.15289898

$$y(x) = -1 + \frac{x}{1\,!}(2) + \frac{x(x-1)}{2\,!}(1)$$

$$= -1 + 2x + \frac{1}{2}x^2 - \frac{1}{2}x$$

$$= -1 + \frac{3}{2}x + \frac{1}{2}x^2$$

4. **From the date given below, find y at $x = 43$.** **QC 80610 MA 6452 NOV 2016**

x	**40**	**50**	**60**	**70**	**80**	**90**
y	**184**	**204**	**226**	**250**	**276**	**304**

Here x values are equally spaced. Also $x = 43$ lies in the beginning of the table. Hence we use Newton's forward difference formula.

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		
70	250		2	
		26		
80	276		2	
		28		
90	304			

Forward formula

$$y(x) = y_0 + \frac{u}{1\,!}\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots \text{,where}$$

$$u = \frac{x - x_0}{h} = \frac{43-40}{10} = 0.3$$

$$y(43) = 184 + \frac{(0.3)}{1!}(20) + \frac{(0.3)(0.3-1)}{2!}(2)$$

$$= 184 + 6 - 0.21$$

$$= 189.79$$

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5. Find the number of students whose weight is between 60 and 70 lbs from the following data using Newton's forward difference interpolation formula. QC 60045 MA 3251 APR 2022

x weight in lbs:	**0-40**	**40-60**	**60-80**	**80-100**	**100-120**
y No. of students:	**250**	**120**	**100**	**70**	**50**

Since x values are given interval, it can be taken as <40, <60, <80, <100, <120. Here x values are equally spaced. We have to find the number of students whose weight is between 60 and 70lbs. It lies in the beginning of the table, hence we use Newton's forward difference formula.

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
< 40	250			
		120		
< 60	370		–20	
		100		–10
< 80	470		–30	
		70		
< 100	540		–20	
		50		
<120	590			

First we find number of students whose weight is < 70 i.e. y(70).

Forward formula

$$y(x) = y_0 + \frac{u}{1\,!}\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \dots,$$

where $u = \frac{x-x_0}{h} = \frac{70-40}{20} = \frac{3}{2}$

$$y(70) = 250 + \frac{\left(\frac{3}{2}\right)}{1\,!}(120) + \frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{2\,!}(-20) + \frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3\,!}(-10)$$

$$= 250+180-7.5+0.62$$

$$= 423$$

Number of students whose weight is between 60 and 70lbs = y(70) – y(60) = 423 – 370 = 53.

6. Find the value of $\tan 45^\circ 15'$ by using Newton's forward difference interpolation formula for

x°	:	**45**	**46**	**47**	**48**	**49**	**50**
$\tan x^\circ$	:	**1**	**1.03553**	**1.07237**	**1.11061**	**1.15037**	**1.19175**

QC 41316 MA 6459 MAY 2018

Here x values are equally spaced. Also $x = 45°15'$ lies in the beginning of the table, hence we use Newton's forward difference formula.

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	1			
		0.03553		
46	1.03553		0.00131	
		0.03684		0.00009
47	1.07237		0.0014	
		0.03824		0.00012
48	1.11061		0.00152	
		0.03976		0.0001
49	1.15037		0.00162	
		0.04138		
50	1.19175			

Forward formula

$$y(x) = y_0 + \frac{u}{1\,!}\Delta y_0 + \frac{u(u-1)}{2\,!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_0 + \ldots \text{,where}$$

$$u = \frac{x - x_0}{h} = \frac{45°15' - 45}{1°} = \frac{15'}{60'} = 0.25$$

$$y(45°15') = 1 + \frac{(0.25)}{1!}(0.03553) + \frac{(0.25)(-0.75)}{2!}(0.00131) + \frac{(0.25)(-0.75)(-1.75)}{3!}(0.00009)$$

$$= 1 + 0.0088825 - 0.0001228 + 0.00000323$$

$$\tan 45°15' = 1.00876$$

Solved Problems in Interpolation by Newton's Backward Difference

11. Give the Newton's backward difference table for **QC 27331 MA 6452 NOV 2015**

x :	**0**	**1**	**2**	**3**
y(x) :	**–1**	**–2**	**–1**	**2**

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	–1			
		–1		
1	–2		2	
		1		0
2	–1		2	
		3		
3	2			

https://doi.org/10.5281/zenodo.15289898

12. **From the date given below, find y at $x = 84$.** **QC 80610 MA 6452 NOV 2016**

x	40	50	60	70	80	90
y	184	204	226	250	276	304

Here x values are equally spaced. Since $x = 84$ lies in the end of the table, we use Newton's backward difference formula.

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		
70	250		2	
		26		
80	276		2	
		28		
90	304			

Backward formula

$$y(x) = y_n + \frac{u}{1\,!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \text{,where}$$

$$u = \frac{x - x_n}{h} = \frac{84-90}{10} = -0.6$$

$$y(84) = 304 + \frac{(-0.6)}{1!}(28) + \frac{(-0.6)(-0.6+1)}{2!}(2)$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

13. **Given:**

x	**:**	**140**	**150**	**160**	**170**	**180**
y	**:**	**3.685**	**4.854**	**6.302**	**8.076**	**10.225**

Find $y(175)$. **QC 27331 MA 6452 NOV 2015**

Here x values are equally spaced. We have to find y at $x = 175$. Since $x = 175$ lies in the end of the table, we use Newton's backward difference formula.

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
140	3.685			
		1.169		
150	4.854		0.279	
		1.448		0.047
160	6.302		0.326	
		1.774		0.049
170	8.076		0.375	
		2.149		
180	10.225			

Backward formula

$$y(x) = y_n + \frac{u}{1\,!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \text{,where}$$

$$u = \frac{x - x_n}{h} = \frac{175-180}{10} = -\frac{1}{2}$$

$$y(175) = 10.225 + \frac{\left(-\frac{1}{2}\right)}{1!}(2.149) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}{2!}(0.375) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)}{3!}(0.049)$$

$$= 10.225 - 1.0745 - 0.04688 - 0.00306$$

$$= 9.10056$$

14. Use Newton's backward difference formula to fit a third degree polynomial for the following data: **QC 20753 MA 6452 NOV 2018**

x	:	**−0.75**	**−0.5**	**−0.25**	**0**
$f(x)$	:	**−0.0718125**	**−0.02475**	**0.3349375**	**1.101**

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
−0.75	−0.0718125			
		0.0470625		
−0.5	−0.02475		0.312625	
		0.3596875		0.09375
−0.25	0.3349375		0.406375	
		0.7660625		
0	1.101			

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Backward formula

$$y(x) = y_n + \frac{u}{1\ !}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots \text{,where} \quad u = \frac{x - x_n}{h} = \frac{x-0}{0.25} = 4x$$

$$y(x) = 1.101 + \frac{(4x)}{1!}(0.7660625) + \frac{(4x\ )(4x+1)}{2!}(0.406375) + \frac{(4x)(4x+1)(4x+2)}{3!}(0.09375)$$

$$= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.5x^2 + 0.125x$$

$$= x^3 + 3.751x^2 + 4.002x + 1.101$$

15. Apply Newton's backward formula to find a polynomial of degree 3.

x	:	**3**	**4**	**5**	**6**
y	:	**6**	**24**	**60**	**120**

QC 72071 MA 6452 MAY 2017

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	6			
		18		
4	24		18	
		36		6
5	60		24	
		60		
6	120			

Backward formula

$$y(x) = y_n + \frac{u}{1\ !}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots \text{,where}$$

$$u = \frac{x - x_n}{h} = \frac{x-6}{1} = x - 6$$

$$y(x) = 120 + \frac{(x-6)}{1!}(60) + \frac{(x-6\ )(x-5)}{2!}(24) + \frac{(x-6\ )(x-5)(x-4)}{3!}(6)$$

$$= 120 + 60x - 360 + 12x^2 - 132x + 360 + x^3 - 15x^2 + 74x - 120$$

$$= x^3 - 3x^2 + 2x$$

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Interpolation for unequal intervals.

Consider the tabulated values of the function $y = f(x)$ for unequally spaced values of $x = x_0, x_1, \ldots, x_n$ as $y = y_0, y_1, y_2, \ldots, y_n$. To determine the polynomial or to determine interpolating values, the following methods are used.

(i) Lagrange's Method (ii) Newton's Divided Difference Method

Lagrange's formula for n pair of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$.

$$y(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \ldots\ldots\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n$$

Note: Lagrange's formula can also be used for equally spaced values of x.

Inverse interpolation is also possible in Lagrange's method. The formula is obtained by interchanging the variables x and y.

The method of finding the value of x for some intermediate value of y is known as inverse interpolation

Solved Problems in Lagrange's Interpolation

1. Obtain Lagrangian interpolation polynomial from the data. **QC 41313 MA 6452 MAY 2018**

x	**0**	**1**	**3**
$f(x)$	**5**	**6**	**14**

Lagrange's formula for the given data is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2.$$

$$y(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}5 + \frac{(x-0)(x-3)}{(1-0)(1-3)}6 + \frac{(x-0)(x-1)}{(3-0)(3-1)}14.$$

$$= \frac{5}{3}\left(x^2-4x+3\right) - 3\left(x^2-3x\right) + \frac{7}{3}\left(x^2-x\right)$$

$$= x^2\left(\frac{5}{3} - 3 + \frac{7}{3}\right) + x\left(\frac{-20}{3} + 9 - \frac{7}{3}\right) + (5)$$

$$= x^2 + 5$$

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2. Using Lagrange's formula, fit a polynomial to the data:

x	-1	0	2	3
y	-8	3	1	12

Hence find y at $x = 1.5$ and $x = 1$. **QC 80610 MA 6452 NOV 2016**

Here x values are not equally spaced, Hence we use Lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

$$y(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$

$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12).$$

$$y(x) = \frac{2}{3}\left(x^3-5x^2+6x\right)+\frac{1}{2}\left(x^3-4x^2+x+6\right)-\frac{1}{6}\left(x^3-2x^2-3x\right)+\left(x^3-x^2-2x\right)$$

$$= x^3\left(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1\right)+x^2\left(-\frac{10}{3}-\frac{4}{2}+\frac{2}{6}-1\right)+x\left(\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2\right)+3$$

$$= 2x^3-6x^2+3x+3$$

$$y(1) = 2-6+3+3 = 2$$

$$y(1.5) = 2(1.5)^3-6(1.5)^2+3(1.5)+3 = 0.75$$

3. Use Lagrange's interpolation formula to fit a polynomial to the following data and hence find $f(2)$. **QC 53252 MA 6459 MAY 2019**

x	0	1	3	4
$f(x)$	-12	0	6	12

Here x values are not equally spaced, Hence we use Lagrange's formula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

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$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0)$$

$$+\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12).$$

$$f(x) = \left(x^3-8x^2+19x-12\right)+\left(-x^3+5x^2-4x\right)+\left(x^3-4x^2+3x\right)$$

$$= x^3\left(1-1+1\right)+x^2\left(-8+5-4\right)+x\left(19-4+3\right)-12$$

$$= x^3-7x^2+18x-12$$

$$f(2)=2^3-7\times 2^2+18\times 2-12=8-28+36-12=4$$

4. Use Lagrange's interpolation formula to find $f(10)$ from the following data:

QC 20753 MA 6452 NOV 2018

x	:	**5**	**6**	**9**	**11**
$f(x)$	:	**12**	**13**	**14**	**16**

Here x values are not equally spaced, Hence we use Lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

$$y(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16).$$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16).$$

$$= \frac{(4)(1)(-1)}{(-1)(-3)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13)$$

$$+\frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16).$$

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$$= \frac{8}{3} - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{46}{3}$$

5. Using Lagrange's interpolation formula, find the polynomial $f(x)$ from the following data:

x :	0	1	4	5
$f(x)$:	4	3	24	39

QC 50782 MA 6452 NOV 2017

Here x values are not equally spaced, Hence we use Lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3.$$

$$y(x) = \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(4) + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24) + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39).$$

$$y(x) = -\frac{1}{5}\left(x^3 - 10x^2 + 29x - 20\right) + \frac{1}{4}\left(x^3 - 9x^2 + 20x\right) - 2\left(x^3 - 6x^2 + 5x\right) + \frac{39}{20}\left(x^3 - 5x^2 + 4x\right)$$

$$= x^3\left(-\frac{1}{5} + \frac{1}{4} - 2 + \frac{39}{20}\right) + x^2\left(2 - \frac{9}{4} + 12 - \frac{39}{4}\right) + x\left(-\frac{29}{5} + 5 - 10 + \frac{39}{5}\right) + 4$$

$$= 2x^2 - 3x + 4$$

Divided Difference

Consider the tabulated values of the function $y = f(x)$ for unequally spaced values of $x = x_0, x_1, \ldots, x_n$ as $y = y_0, y_1, y_2, \ldots., y_n$. To determine the polynomial or to determine interpolating values, the following difference are used.

Divided Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, x_1)$		
x_1	y_1		$\frac{f(x_1,x_2) - f(x_0,x_1)}{x_2 - x_0} = f(x_0, x_1, x_2)$	
		$\frac{y_2 - y_1}{x_2 - x_1} = f(x_1, x_2)$		$\frac{f(x_1,x_2,x_3) - f(x_0,x_1,x_2)}{x_3 - x_0} = f(x_0, x_1, x_2, x_3)$
x_2	y_2		$\frac{f(x_2,x_3) - f(x_1,x_2)}{x_3 - x_1} = f(x_1, x_2, x_3)$	

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x_3	y_3	$\frac{y_3-y_2}{x_3-x_2}=f(x_2,x_3)$		

Newton's Divided Difference Interpolation

1. State Newton's divided difference interpolation formulae. **QC 53250 MA 6452 MAY 2019**

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)............$$

2. Find first and second divided differences with arguments a, b, c of the function $f(x)=\frac{1}{x}$.

QC 41316 MA 6459 MAY 2018

$$If \quad f(x)=\frac{1}{x}, \quad f(a)=\frac{1}{a}$$

$$f(a,b)=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}=\frac{\frac{a-b}{ab}}{b-a}=-\frac{a-b}{ab}.\frac{1}{a-b}=-\frac{1}{ab}$$

$$f(a,b,c)=\frac{f(b,c)-f(a,b)}{c-a}=\frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a}=\frac{-a+c}{abc}\left(\frac{1}{c-a}\right)=\frac{1}{abc}$$

3 If $f(x)=\frac{1}{x^2}$, find the divided difference $f(a,b)$ and $f(a,b,c)$. **QC 60045 MA 3251 APR 2022**

$$If \quad f(x)=\frac{1}{x^2}, \quad f(a)=\frac{1}{a^2}$$

$$f(a,b)=\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b^2-a^2}=\frac{\frac{a^2-b^2}{a^2b^2}}{b^2-a^2}=-\frac{1}{a^2b^2}$$

$$f(a,b,c)=\frac{f(b,c)-f(a,b)}{c^2-a^2}=\frac{-\frac{1}{b^2c^2}+\frac{1}{a^2b^2}}{c^2-a^2}=\frac{1}{a^2b^2c^2}\left(\frac{c^2-a^2}{c^2-a^2}\right)=\frac{1}{a^2b^2c^2}$$

4. State any two properties of divided differences. **QC 80610 MA 6452 NOV 2016**

- Divided difference operator is linear
- Divided differences of all orders are symmetrical in their arguments

5. Write the difference between divided differences and forward differences.

$$\Delta f(x_0)=f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{\Delta f(x_0)}{h}$$

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In general

$$\Delta^n f(x_0) = \frac{\Delta f(x_0)}{n!\, h^n}$$

6. Find the third divided differences of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11.

QC 20753 MA 6452 NOV 2018

Given $f(x) = x^3 + x + 2$ and hence

$f(1) = 1^3 + 1 + 2 = 4$ $\quad\quad f(3) = 3^3 + 3 + 2 = 32$

$f(6) = 6^3 + 6 + 2 = 224$ $\quad\quad f(11) = 11^3 + 11 + 2 = 1344$

Here x values are not equally spaced, Newton's Divided Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		$\frac{32-4}{3-1} = 14$		
3	32		$\frac{64-14}{6-1} = 10$	
		$\frac{224-32}{6-3} = 64$		$\frac{20-10}{11-1} = 1$
6	224		$\frac{224-64}{11-3} = 20$	
		$\frac{1344-224}{11-6} = 224$		
11	1344			

7. Find the divided difference of $f(x) = x^3 + x + 2$ for the arguments $x = 1, 3, 6$.

QC 20817 MA 8452 APR 2022

Here x **: 1 3 6**
$f(x)$ **: 4 32 224**

Here x values are not equally spaced, Hence Newton's Divided Difference table is as follows:

x	y	Δy	$\Delta^2 y$
1	4		
		14	
3	32		10

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		64	
6	224		

8. **Find the divided difference table for the following data.** **QC 41313 MA 6452 MAY 2018**

x	**2**	**5**	**10**
y	**5**	**29**	**109**

Newton's Divided Difference table.

x	y	Δy	$\Delta^2 y$
2	5		
		8	
5	29		1
		16	
10	109		

9. **Given:**

x	:	**0**	**2**	**3**	**4**	**7**	**9**
y	:	**4**	**26**	**58**	**112**	**466**	**922**

Find $y(10),\ y'(6)$ **using Newton's divided difference formula.** **QC 27331 MA 6452 NOV 2015**

Here x values are not equally spaced, Hence we find the polynomial satisfying the values, then using it we can find $y'(6)$ *and* $y(10)$ by Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	4			
		11		
			7	
2	26			
		32		1
3	58			
			11	
		54		1
4	112			
			16	
		118		1
7	466			
			22	
		228		

9	922			

To find the polynomial

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)............$$

$$= 4 + (x-0)(11) + (x-0)(x-2)(7) + (x-0)(x-2)(x-4)(1)$$

$$f(x) = 4 + 11x + 7x^2 - 14x + x^3 - 6x^2 + 8x$$

$$= x^3 + x^2 + 5x + 4$$

$$f(10) = (10)^3 + (10)^2 + 5(2) + 4 = 1114$$

$$f'(x) = 3x^2 + 2x + 5$$

$$f'(6) = 3(6)^2 + 2(6) + 5 = 125$$

10. From the following table find $f(x)$ using Newton's interpolation formula.

QC 53250 MA 6452 MAY 2019

x	**1**	**2**	**7**	**8**
$f(x)$	**1**	**5**	**5**	**4**

Here x values are not equally spaced, Hence we use Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		4		
2	5		$-\frac{4}{6}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		-1		
8	4			

To find the polynomial

$$f(x)= f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)\ldots\ldots\ldots\ldots$$

$$=1+(x-1)(4)+(x-1)(x-2)\left(-\frac{4}{6}\right)+(x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$$f(x)=1+4x-4-\frac{4}{6}\left(x^2-3x+2\right)+\frac{1}{14}\left(x^3-10x^2+23x-14\right)$$

$$=\frac{1}{14}x^3-\frac{29}{21}x^2+\frac{107}{14}x-\frac{16}{3}$$

10. **If** $f(0)=1,\ f(1)=4,\ f(3)=40$ **and** $f(4)=85,$ **find** $f(x)$ **that satisfies this data using Newton's divided difference formula. Hence, find** $f(5)$**.** **QC 60045 MA 3251 APR 2022**

Here

x **: 0 1 3 4** $f(x)$ **: 1 4 40 85**

Here x values are not equally spaced, Hence we use Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		3		
1	4		5	
		18		1
3	40		9	
		45		
4	85			

$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$............

$= 1 + (x-0)(3) + (x-0)(x-1)(5) + (x-0)(x-1)(x-3)(1)$

$f(x) = 1 + 3x + 5x^2 - 5x + x^3 - 3x^2 - x^2 + 3x$

$= x^3 + x^2 + x + 1$

$f(5) = 5^3 + (5)^2 + 5 + 1$

$= 125 + 25 + 5 + 1$

$= 156$

11. **Given** $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8181$, $\log_{10} 659 = 2.8189$ *and* $\log_{10} 661 = 2.8202$. **Find the value of** $\log_{10} 656$ **using Newton's divided difference formula. QC 72071 MA 6452 MAY 2017**

Given values are tabulated as

x	654	658	659	661
$y = \log_{10} x$	2.8156	2.8181	2.8189	2.8202

Here x values are not equally spaced. We have to find y when $x = 656$. Hence we use Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
654	2.8156			
		0.000625		
			0.000035	
658	2.8181			
		0.0008		0.000012
659	2.8189			
			-0.00005	
		0.00065		
661	2.8202			

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$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \ldots\ldots\ldots\ldots$$

$$= 2.8156 + (x - 654)(0.000626) + (x - 654)(x - 658)\left(0.000035\right) + (x - 654)(x - 658)(x - 659)\left(0.000012\right)$$

$$f(656) = 2.8156 + (656 - 654)(0.000626) + (656 - 654)(656 - 658)\left(0.000035\right) + (656 - 654)(656 - 658)(656 - 659)\left(0.000012\right)$$

$$= 2.8156 + 0.001252 - 0.00014 + 0.000144$$

$$= 2.8169$$

12. From the following values, find $f(x)$ and hence find $f(6)$ by Newton's divided difference formula.

x :	**1**	**2**	**7**	**8**
$f(x)$:	**1**	**5**	**5**	**4**

QC 50782 MA 6452 NOV 2017

Here x values are not equally spaced, Hence we use Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		4		
2	5		$-\frac{4}{6}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		-1		
8	4			

To find the polynomial

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) \ldots\ldots\ldots\ldots$$

$$= 1 + (x-1)(4) + (x-1)(x-2)\left(-\frac{4}{6}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$$f(x) = 1 + 4x - 4 - \frac{4}{6}\left(x^2 - 3x + 2\right) + \frac{1}{14}\left(x^3 - 10x^2 + 23x - 14\right)$$

$$= \frac{1}{14}x^3 - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{3}$$

$$f(6) = \frac{1}{14}(6)^3 - \frac{29}{21}(6)^2 + \frac{107}{14}(6) - \frac{16}{3}$$

$$= 6.238$$

13. **Find $y'(1)$, if**

x :	**−1**	**0**	**2**	**3**
$y(x)$:	**−8**	**3**	**1**	**12**

QC 57506 MA 6452 MAY 2016

Here x values are not equally spaced, Hence we find the polynomial satisfying the values, then using it we can find $y'(1)$ by Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
−1	−8			
		11		
0	3		−4	
		−1	4	**2**
2	1			
		11		
3	12			

To find the polynomial

$f(x)= f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$............

$= -8 + (x+1)(11) + (x+1)(x-0)(-4) + (x+1)(x-0)(x-2)(2)$

$f(x) = -8+11x+11-4(x^2+x)+2(x^3-x^2-2x)$

$= 2x^3-6x^2+3x+3$

$f'(x) = 6x^2-12x+3$

$f'(1) = 6(1)^2-12(1)+3$

$= -3$

14. Find $f'(10)$ from the following data:

x :	**3**	**5**	**11**	**27**	**34**
$f(x)$:	**−13**	**23**	**899**	**17315**	**35606**

QC 27335 MA 6459 NOV 2015

Here x values are not equally spaced, Hence we find the polynomial satisfying the values, then using it we can find $f'(10)$ by Newton's Divided Difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
3	−13			
		$\frac{23+13}{5-3}=18$		
			$\frac{146-18}{11-3}=16$	
5	23			
				$\frac{40-16}{27-3}=1$
		$\frac{899-23}{11-5}=146$		
			$\frac{1026-146}{27-5}=40$	
11	899			
		$\frac{17315-899}{27-11}=1026$		
				$\frac{69-40}{34-5}=1$
27	17315			
			$\frac{2613-1026}{34-11}=69$	

https://doi.org/10.5281/zenodo.15289898

34	35606	$\frac{35606-17315}{34-27}=2613$		

To find the polynomial

$$f(x)= f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)\ldots\ldots\ldots\ldots$$

$$=-13+(x-3)(18)+(x-3)(x-5)(16)+(x-3)(x-5)(x-11)(1)$$

$$f(x)=-13+18x-54+\left(16x^2-128x+240\right)+\left(163x^3-304x^2+1648x-2640\right)$$

$$=16x^3-288x^2+1538x-2467$$

$$f'(x)=48x^2-576x+1538$$

$$f'(10)=48(10)^2-576(10)+1538$$

$$=578$$

Numerical Differentiation

Instead of functional form of $f(x)$, suppose a set of values of x and y are given. Assume that the values of x are equally spaced. Then the process of finding its derivatives is known as numerical differentiation.

Differentiate Newton's forward difference formula w.r.t x, we have

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \text{ ,where } u=\frac{x-x_0}{h}$$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx} = \left[\frac{1}{1!}\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \ldots\right].\frac{1}{h}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots\right]\ldots..(1)$$

$$\frac{d^2y}{dx^2}=\frac{d}{du}\left(\frac{dy}{dx}\right).\frac{du}{dx} = \frac{1}{h}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \ldots\right].\frac{1}{h}$$

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$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \dots \right] \quad \dots\dots(2)$$

When $x = x_0$, then $u = 0$. Therefore equation (1) and (2) becomes

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots\right]$$

Similarly, we can derive differentiation formulas from Newton's backward formula as follows:

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \left(\frac{6u^2+18u+11}{12}\right)\nabla^4 y_n + \dots\right]$$

When $x = x_n$, then $u = 0$. Therefore the above formulas becomes

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right]$$

1. When can numerical differentiation be used.

i. For the unknown function $y = f(x)$ with the numerical values of x and y

ii. If the ordinary differentiation is complicated

iii. the differences of y of some order are constant

2. What are the two types of errors involving in the computation of derivatives.

i. Round-off error ii. Truncation error

3. Find the error in the derivative of $f(x) = \cos x$ by computing directly and using the approximation $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h}$ at $x = 0.8$ choosing $h = 0.01$.

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$f(x) = \cos x$ $\qquad$ $f(x+h) = f(0.8+0.01) = \cos 0.89 = 0.689498$

$f'(x) = -\sin x$ $\qquad$ $f(x-h) = f(0.8-0.01) = \cos 0.79 = 0.703845$

$f'(0.8) = -\sin 0.8 = -0.71735609$ ....(i) $\qquad$ $\dfrac{f(x+h) - f(x-h)}{2h} = -0.717365782$ ....(ii)

$\therefore$ Error = (ii) – (i) = 0.00000969

Derivatives by Forward Difference Formula

1 **Find the gradient of the road at the initial point of the elevation above a datum line of seven points of road which are given below:** **QC 80221 MA 8491 MAY 2019**

x :	**0**	**300**	**600**	**900**	**1200**	**1500**	**1800**
$f(x)$:	**135**	**149**	**157**	**183**	**201**	**205**	**193**

We know that gradient is the slope i.e. $\dfrac{dy}{dx}$. Since the initial point $x = 0$ lies at the beginning of the table, we use forward formula for derivatives. Also x values are equally spaced with $h = 300$

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	135					
		14				
300	149		–6			
		8		24		
600	157		18		–50	
		26		–26		70
900	183		–8		20	
		18		–6		–16
1200	201		–14		4	
		4		–2		
1500	205		–16			
		–12				
1800	193					

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$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - ....\right\}$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{300}\left\{14 - \frac{1}{2}(-6) + \frac{1}{3}(24) - \frac{1}{4}(-50) + \frac{1}{5}(70)\right\} = 0.1716$$

2 **The table gives below reveals the velocity v of a body during the time 't' specified. Find its acceleration at $t = 1.1$.** **QC 80221 MA 8491 MAY 2019**

t :	**1**	**1.1**	**1.2**	**1.3**	**1.4**
v :	**43.1**	**47.7**	**32.1**	**56.4**	**60.8**

We know that acceleration $= \frac{dv}{dt}$. So we have to find first derivative of v at $t = 1.1$

Since $t = 1.1$ lies at the beginning of the table, we use forward formula for derivatives. Also t values are equally spaced with $h = 0.1$

Forward difference Table:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1	43.1				
		4.6			
1.1	47.7		−20.2		
		−15.6		60.1	
1.2	32.1		39.9		−119.9
		24.3		−59.8	
1.3	56.4		−19.9		
		4.4			
1.4	60.8				

Acceleration

$$\frac{dv}{dt} = \frac{1}{h}\left\{\Delta v_0 + \frac{2p-1}{2}\Delta^2 v_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 v_0 + \frac{4p^3 - 18p^2 + 22p - 6}{24}\Delta^4 v_0 + ...\right\} \quad where \;\; p = \frac{t - t_0}{h} = \frac{1.1 - 1}{0.1} = 1$$

$$\left(\frac{dv}{dt}\right)_{t=1.1} = \frac{1}{h}\left\{\Delta v_0 + \frac{1}{2}\Delta^2 v_0 - \frac{1}{6}\Delta^3 v_0 + \frac{1}{12}\Delta^4 v_0 + ...\right\}$$

$$\left(\frac{dv}{dt}\right)_{t=1.1} = \frac{1}{0.1}\left\{4.6 + \frac{1}{2}(-20.2) - \frac{1}{6}(60.1) + \frac{1}{12}(-119.9)\right\} = -255.1$$

3. **Find the first two derivatives of the function at $x = 1.5$ from the table below using Newton's forward formula:** **QC 60045 MA 3251 APR 2022**

x :	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
y :	**3.375**	**7.0**	**13.625**	**24**	**38.875**	**59**

Since $x = 1.5$ lies at the beginning of the table, we use forward formula for derivatives. Also x values are equally spaced with $h = 0.5$

Forward difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2	7		3		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3	24		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4	59				

First Derivative

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right\}$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left\{3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right\} = 4.75$$

Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - .....\right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = \frac{1}{(0.5)^2}\{3 - 0.75\} = 9$$

4. **Find the value of** $\cos(1.74)$ **, using suitable formula from the following data.**

QC 20753 MA 6452 NOV 2018

x :	1.7	1.74	1.78	1.82	1.86
$\sin(x)$:	0.9916	0.9857	0.9781	0.9691	0.9584

Here x values are equally spaced. Given $y = \sin x$ and we have to find $y' = \cos(1.74)$. Also $x = 1.74$ lies in the beginning of the table, hence we use Newton's forward difference formula.

x	y	Δy	$\Delta^2 y$
1.7	0.9916		
		−0.0059	
1.74	0.9857		−0.0017
		−0.0076	
1.78	0.9781		−0.0014
		−0.009	
1.82	0.9691		−0.0017
		−0.0107	

1.86	0.9584		

$$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{3p^2-6p+2}{6}\Delta^3 y_0+\ldots\ldots\right\} \quad where \ \ p=\frac{x-x_0}{h}=\frac{1.74-1.7}{0.04}=1$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{1}{2}\Delta^2 y_0-\frac{1}{6}\Delta^3 y_0+\ldots\ldots\right\}$$

$$\left[\frac{dy}{dx}\right]_{x=1.74}=\frac{1}{0.04}\left\{-0.0059+\frac{1}{2}(-0.0017)\right\}$$

$$=-0.16875$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\Delta^2 y_0+(p-1)\Delta^3 y_0+\ldots\ldots\right\} \quad where \ \ p=\frac{x-x_0}{h}=\frac{1.74-1.7}{0.04}=1$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\Delta^2 y_0+(0)\Delta^3 y_0+\ldots\ldots\right\}$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.74}=\frac{1}{(0.04)^2}\left\{-0.0017\right\}$$

$$=-1.0625$$

5. Find the value of $\sin\left(18^\circ\right)$ from the following table, using numerical differentiation based on Newton's forward interpolation formula. **QC 53252 MA 6459 MAY 2019**

x° :	0	10	20	30	40
$\cos\left(x^\circ\right)$:	1	0.9848	0.9397	0.866	0.766

Here x values are equally spaced. Given $y=\cos x$ and we have to find $y'=-\sin(18)$. Also $x=18$ lies in the beginning of the table, hence we use Newton's forward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		−0.0152		
10	0.9848		−0.0299	
		−0.0451		0.0013
20	0.9397		−0.0286	
		−0.0737		0.0023

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30	0.866		−0.0263	
		−0.1		
40	0.766			

$$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{3p^2-6p+2}{6}\Delta^3 y_0+\ldots\ldots\right\}\ \text{where}\ p=\frac{x-x_0}{h}=\frac{18-0}{10}=1.8$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{2\times1.8-1}{2}\Delta^2 y_0+\frac{3\times1.8^2-6\times1.8+2}{6}\Delta^3 y_0+\ldots\ldots\right\}$$

$$\left[\frac{dy}{dx}\right]_{x=18}=\frac{180}{10\pi}\{(-0.0152)+(1.3)(-0.0299)+(0.1533)(0.0013)\}$$

$$=-0.3086$$

Given $y=\cos x$ and hence $\frac{dy}{dx}=-\sin x$

Therefore $-\sin18^\circ=-0.3086$

$$\sin18^\circ=0.3086$$

Derivatives by Backward Difference Formula

6. Specify the Newton's backward difference formulae for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. QC 57506 MA 6452 MAY 16

$$\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left\{\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots\right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left\{\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+\ldots\ldots\right\}$$

7. Write Newton's backward difference formula to find the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=x_n$.
QC 20753 MA 6452 NOV 2018

https://doi.org/10.5281/zenodo.15289898

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left\{\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + ...\right\} \text{ where } p = \frac{x-x_n}{h}$$

$$= \frac{1}{h}\left\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + ...\right\} \text{when } p = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left\{\nabla^2 y_n + (p+1)\nabla^3 y_n + ...\right\} \text{ where } p = \frac{x-x_n}{h}$$

$$= \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + ...\right\} \text{ when } p = 0$$

8. The population y (in thousands) of a certain town is given below. Find the rate of growth of the population in 1961. **QC 20817 MA 8452 APR 2022**

x:	**1931**	**1941**	**1951**	**1961**	**1971**
y:	**40.62**	**60.8**	**79.95**	**103.56**	**132.65**

Rate of growth of population at 1961 is given by $\frac{dy}{dx}$ at $x = 1961$.

Since $x = 1961$ lies at the end of the table, we use backward formula for derivatives. Also x values are equally spaced with $h = 10$.

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1931	40.62				
		20.18			
1941	60.8		−1.03		
		19.15		14.49	
1951	79.95		13.46		−40.47
		32.61		−25.98	
1961	103.56		−12.52		
		29.09			
1971	132.65				

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{2p+1}{2}\nabla^2 y_n+\frac{3p^2+6p+2}{6}\nabla^3 y_n+......\right\} \quad where \ \ p=\frac{x-x_n}{h}=\frac{1961-1971}{10}=-1$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n-\frac{1}{2}\nabla^2 y_n-\frac{1}{6}\nabla^3 y_n+......\right\}$$

$$\left[\frac{dy}{dx}\right]_{x=1961}=\frac{1}{10}\left\{29.9-\frac{1}{2}(-12.52)-\frac{1}{6}(-25.98)\right\}$$

$$= 4.049$$

9. Find the first derivative of $f(x)$ at $x=0.4$ from the following table. QC 41313 MA 6452 MAY 2018

x	**0.1**	**0.2**	**0.3**	**0.4**
$f(x)$	**1.10517**	**1.2214**	**1.34986**	**1.49182**

Here x values are equally spaced. Also $x=0.4$ lies in the end of the table, hence we use Newton's backward difference formula.

Backward difference Table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0.1	1.10517			
		0.11623		
0.2	1.2214		0.01223	
		0.12846		0.00127
0.3	1.34986		0.0135	
		0.14196		
0.4	1.49182			

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{2p+1}{2}\nabla^2 y_n+\frac{3p^2+6p+2}{6}\nabla^3 y_n+......\right\} \quad where \ \ p=\frac{x-x_n}{h}=\frac{0.4-0.4}{0.1}=0$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+......\right\}$$

$$\left[\frac{dy}{dx}\right]_{x=0.4}=\frac{1}{0.1}\left\{0.14196+\frac{1}{2}(0.0135)+\frac{1}{3}(0.00127)\right\}$$

$$= 1.4913$$

10. Find the first two derivatives of $x^{\frac{1}{3}}$ *at* $x=56$, given the table below: QC 41313 MA 6452 MAY 2018

x	**50**	**51**	**52**	**53**	**54**	**55**	**56**

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$y=x^{\frac{1}{3}}$	**3.684**	**3.7084**	**3.7325**	**3.7563**	**3.7798**	**3.803**	**3.8259**

Here x values are equally spaced. Also $x=56$ lies in the end of the table, hence we use Newton's backward difference formula.

Backward difference Table:

x	y	∇y	$\nabla^2 y$
50	3.684		
		0.0244	
51	3.7084		−0.0003
		0.0241	
52	3.7325		−0.0003
		0.0238	
53	3.7563		−0.0003
		0.0235	
54	3.7798		−0.0003
		0.0232	
55	3.803		−0.0003
		0.0229	
56	3.8259		

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{2p+1}{2}\nabla^2 y_n+\frac{3p^2+6p+2}{6}\nabla^3 y_n+......\right\}\ where\ p=\frac{x-x_n}{h}=\frac{56-56}{1}=0$$

$$\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+......\right\}$$

$$\left[\frac{dy}{dx}\right]_{x=56}=\frac{1}{1}\left\{0.0229+\frac{1}{2}(-0.0003)\right\}$$

$$=0.02275$$

https://doi.org/10.5281/zenodo.15289898

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + (p+1)\nabla^3 y_n + ...\right\} \quad where \quad p = \frac{x - x_n}{h} = \frac{56-56}{1} = 0$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + ...\right\}$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=56} = \frac{1}{1}\{-0.0003\}$$

$$= -0.0003$$

11. **A rod is rotating a plane. The following table gives the angle θ with respect to time t seconds.**

t :	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
θ :	**0**	**0.12**	**0.49**	**1.12**	**2.02**	**3.2**

Calculate the angular velocity and angular acceleration of the rod when $t = 0.6$ seconds.

QC 72071 MA 6452 MAY 2017

Here t values are equally spaced with $h = 0.2$. Also $t = 0.6$ lies in the end of the table, hence we use Newton's backward difference formula.

t	θ	∇θ	∇²θ	∇³θ	∇⁴θ
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.9		0.01	
0.8	2.02		0.28		
		1.18			
1	3.2				

First Derivative (Angular Velocity)	**Second Derivative (Angular Acceleration)**
	$\left(\frac{d^2\theta}{dt^2}\right)_{t=t_n} = \frac{1}{h^2}\left\{\nabla^2\theta_n + \nabla^3\theta_n + \frac{11}{12}\nabla^4\theta_n + ...\right\}$ $\left(\frac{d^2\theta}{dt^2}\right)_{x=6} = \frac{1}{0.04}\{0.26 + 0.01\}$ $= 6.75$

https://doi.org/10.5281/zenodo.15289898

$\left(\frac{d\theta}{dt}\right)_{t=t_n} = \frac{1}{h}\left\{\nabla\theta_n + \frac{1}{2}\nabla^2\theta_n + \frac{1}{3}\nabla^3\theta_n + \frac{1}{4}\nabla^4\theta_n + ...\right\}$ $\left(\frac{d\theta}{dt}\right)_{t=6} = \frac{1}{0.2}\left\{0.63 + \frac{1}{2}0.26 + \frac{1}{3}0.01\right\}$ $= \frac{1}{0.2}\{0.63 + 0.13 + 0.0033\}$ $= 3.816$	

Numerical Integration – Trapezoidal Rule

Newton's Cotes quadrature formula: $\int_{x_0}^{x_n} y dx = nh\left\{y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...\right\}$

It is obtained from the Newton Cote's formula by putting $n = 1$ i.e. assuming that there are only two paired values of x & y or that the interpolating polynomial is linear.

Geometrical meaning of trapezoidal rule.

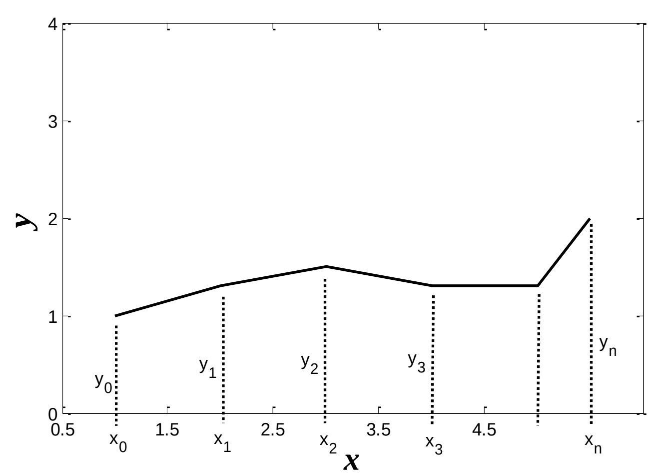
Geometrically, if the ordered pairs (x_i, y_i), $i = 0, 1, 2, ..., n$ are plotted, and if any two consecutive points are joined by straight lines we get the figure as shown.

The area between $f(x)$, x-axis and ordinates $x = x_0$ and $x = x_n$ is approximated to the sum of the trapeziums as shown here.

Here

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1)$$

$$= Area\ of\ the\ first\ Trapezium$$



1. **Write Trapezoidal rule.** **QC 53250 MA 6452 MAY 2019**

Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\}$ where $h = \frac{x_n - x_0}{n}$

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2. Why trapezoidal rule is so called.

The area between $f(x)$, $x-$axis and ordinates $x = x_0$ and $x = x_n$ is approximated to the sum of trapeziums. (basic principle of Trapezoidal rule)

2. What is the order of error in Trapezoidal rules? **QC 50782 MA 6452 NOV 2017**

The error in the Trapezoidal rule is of the order of h^2.

3. Write down the errors in Trapezoidal rule of numerical integration. **QC 57506 MA 6452 MAY 16**

Error in Trapezoidal rule $= -\frac{1}{12}\left(x_n - x_0\right)h^2 f''(x)$ i.e. the truncation error is O(h²)

4. Evaluate $\int_1^2 \frac{1}{1+x^2}dx$ taking $h = 0.2$ using Trapezoidal rule. **QC 27331 MA 6452 NOV 2015**

By Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})\}$ where $h = \frac{x_n - x_0}{n}$

Given $h = \frac{1}{4} = 0.2$

x	1	1.2	1.4	1.6	1.8	2
$y = \frac{1}{1+x^2}$	0.5	0.4098	0.3378	0.2808	0.2358	0.2

$$\int_1^2 \frac{1}{1+x^2}\ dx = \frac{h}{2}\{(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\}$$

$$\int_1^2 \frac{1}{1+x^2}dx = \frac{0.2}{2}\{(0.5 + 0.2) + 2(0.4098 + 0.3378 + 0.2808 + 0.2358)\}$$

$$= \frac{0.2}{2}\{0.7 + 2.5284\}$$

$$= 0.3228$$

5. Evaluate $\int_{-3}^{3} x^4\ dx$ dividing the range of integration into 8 equal parts using Trapezoidal rule. Also compare the results with actual integration. **QC 80610 MA 6452 NOV 2016**

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Let $h = \frac{3-(-3)}{8} = 0.75$

x	-3	-2.25	-1.5	-0.75	0	0.75	1.5	2.25	3
$y = x^4$	81	25.63	5.06	0.316	0	0.316	5.06	25.63	81

By Trapezoidal rule, $\int_{-3}^{3} x^4\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})\}$

$$\int_{-3}^{3} x^4\ dx = \frac{h}{2}\{(y_0 + y_8) + 2(y_1 + y_2 + \dots\dots + y_7)\}$$

$$= \frac{0.75}{2}\{(81+81) + 2(25.63 + 5.06 + 0.316 + 0.316 + 5.06 + 25.63)\}$$

$$= \frac{0.75}{2}\{162 + 124.024\}$$

$$= 107.259$$

Actual Integration:

$$\int_{-3}^{3} x^4\ dx = 2\int_{0}^{3} x^4\ dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2}{5}(243) = 97.2$$

Comparing, the actual value of integration with the Trapezoidal integration values, we see that the difference is significant.

6. **Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Trapezoidal rule with $h = \frac{1}{5}$. Hence obtain an approximate value of π.** **QC 53250 MA 6452 MAY 2019**

By Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})\}$ where $h = \frac{x_n - x_0}{n}$

Given $h = \frac{1}{5} = 0.2$

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1	0.9615	0.8621	0.7353	0.6097	0.5

$$\int_0^1 \frac{1}{1+x^2}\ dx = \frac{h}{2}\{(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.2}{2}\{(1+0.5)+2(0.9615+0.8621+0.7353+0.6097)\}$$

$$= \frac{0.2}{2}\{1.5+6.3372\}$$

$$= 0.75372$$

By actual Integration, we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1$$

$$0.75372 = \tan^{-1}1 - \tan^{-1}0$$

$$0.75372 = \frac{\pi}{4}$$

$$3.01488 \approx \pi$$

7. Evaluate $\int_0^2 e^x\,dx$ by using Trapezoidal rule, taking 6 sub intervals. **QC 41313 MA 6452 MAY 2018**

By Trapezoidal rule, $\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+........+y_{n-1})\}$ where $h = \frac{x_n - x_0}{n}$

$$h = \frac{x_n - x_0}{n} = \frac{2-0}{6} = \frac{1}{3}$$

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$y = e^x$	1	1.3956	1.9477	2.7182	3.7936	5.2944	7.389

$$\int_0^1 e^2\,dx = \frac{h}{2}\{(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\}$$

$$\int_0^1 e^x\,dx = \frac{1}{3\times 2}\{(1+7.389)+2(1.3956+1.9477+2.7182+3.7936+5.2944)\}$$

$$= \frac{1}{6}\{8.389+30.299\}$$

$$= 6.448$$

8 Apply Trapezoidal method to evaluate $I = \int_0^1 e^{x^2}\,dx$, taking $h = 0.2$. **QC 60045 MA 3251 APR 2022**

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By Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\}$ and given $h = 0.2$

x	0	0.2	0.4	0.6	0.8	1
$y = e^{x^2}$	1	1.0408	1.1735	1.4333	1.8964	2.7182

$$\int_0^1 e^{x^2}\ dx = \frac{h}{2}\{(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\}$$

$$\int_0^1 e^{x^2}\ dx = \frac{0.2}{2}\{(1 + 2.7182) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964)\}$$

$$= \frac{0.2}{2}\{3.7182 + 11.088\}$$

$$= 1.4806$$

9 **Evaluate** $\int_0^1 \frac{1}{1+x} dx$ **using Trapezoidal rule. QC 20817 MA 8452 APR 2022**

By Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\}$ where $h = \frac{x_n - x_0}{n}$

Let number of intervals $n = 5$. Then $h = \frac{1-0}{5} = 0.2$

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x}$	1	0.8333	0.7142	0.625	0.555	0.5

$$\int_0^1 \frac{1}{1+x}\ dx = \frac{h}{2}\{(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\}$$

$$\int_0^1 \frac{1}{1+x} dx = \frac{0.2}{2}\{(1 + 0.5) + 2(0.8333 + 0.7142 + 0.625 + 0.555)\}$$

$$= \frac{0.2}{2}\{1.5 + 5.455\}$$

$$= 0.6955$$

10. **Evaluate** $\int_0^1 \frac{1}{1+x^2}dx$ **using Trapezoidal rule with** $h=\frac{1}{4}$ **and hence, compute an approximate value of** π **.** **QC 20817 MA 8452 APR 2022**

(i) By Trapezoidal rule, $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+........+y_{n-1})\}$ where $h=\frac{x_n-x_0}{n}$

Given $h=\frac{1}{4}=0.25$

x	0	0.25	0.5	0.75	1
$y=\frac{1}{1+x^2}$	1	0.9411	0.8	0.64	0.5

$$\int_0^1 \frac{1}{1+x^2}\ dx = \frac{h}{2}\{(y_0+y_4)+2(y_1+y_2+y_3)\}$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{2}\{(1+0.5)+2(0.9411+0.8+0.64)\}$$

$$= \frac{0.25}{2}\{1.5+4.7622\}$$

$$= 0.7827$$

By actual Integration, we have

$$\int_0^1 \frac{1}{1+x^2}\ dx = \left[\tan^{-1} x\right]_0^1$$

$$0.7827 = \tan^{-1}1 - \tan^{-1}0$$

$$0.7827 = \frac{\pi}{4}$$

$$3.1308 \approx \pi$$

11. **Evaluate** $\int_0^1 \frac{1}{1+x}\ dx$ **, using Trapezoidal rule with** $h=0.125$ **and compare the values with exact value.** **QC 20753 MA 6452 NOV 2018**

Given $h=0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1

$y=\frac{1}{1+x}$	1	0.888	0.8	0.7272	0.666	0.6153	0.5714	0.533	0.5

By Trapezoidal rule, $\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+........+y_{n-1})\}$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{2}\{(y_0+y_8)+2(y_1+y_2+y_3+.....+y_7)\}$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{0.125}{2}\{(1+0.5)+2(0.888+0.8+0.7272+0.666+0.6153+0.5714+0.533)\}$$

$$=\frac{0.125}{2}\{1.5+9.6018\}$$

$$=0.6938$$

Actual Integration value

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\log_e(1+x)\right]_0^1 = \log_e 2 - \log_e 1 = 0.6931$$

Numerical Integration – Simpson's $\frac{1}{3}$ Rule

Simpson's $\frac{1}{3}$ rule $\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+........+y_{n-2})+4(y_1+y_3+y_5+.......+y_{n-1})\}$

where $h=\frac{x_n-x_0}{n}$, $n=$number of intervals (multiple of 2)

It is obtained from the Newton Cote's formula by putting $n=2$ i.e. assuming that there are only three paired values of x & y or that the interpolating polynomial is a second degree polynomial.

Note:

1. It gives exact answer if the integrand is a polynomial of degree 2.
2. Since the formula contains both y_0 and y_n, it is called closed formula.

1. Compare Trapezoidal rule with Simpson's $\frac{1}{3}$ rule. **QC 27331 MA 6452 NOV 2015**

Trapezoidal Rule	Simpson's $\frac{1}{3}$ rule
It gives least accurate value	It gives more accurate value
Value is accurate if y is linear	Value is accurate it y is quadratic
No restriction on number of intervals	Number of intervals must be even

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2. **What is the order of error in Simpson's one-third rule? QC 50782 MA 6452 NOV 2017**

The error in the Simpson's rule is of the order of h^4

3. **Write down the errors in Simpson's rule of numerical integration., QC 57506 MA 6452 MAY 2016**

Error in Simpson's rule $= -\frac{1}{180}\left(x_{2n} - x_0\right)h^4 f''''(x)$

4. **What is the difference between the values obtained by evaluating $\int_0^2 x^2 - x\,dx$ directly and by Simpson's rule with 4 intervals.**

By Direct Integration: $\int_0^2 x^2 - x\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2 = \frac{2}{3}$

By Simpson's $\frac{1}{3}$ Rule

$$\int_0^2 x^2 - x\,dx = \frac{h}{3}\{(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\} \quad = \frac{0.5}{3}\{(0+2) + 2(0) + 4(-0.25 + 0.75)\} = \frac{2}{3}$$

5. **Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's $\frac{1}{3}$ rule with $h = \frac{1}{4}$.**
Hence, compute an approximate value of π . QC 20817 MA 8452 APR 2022

(i) Simpson's $\frac{1}{3}$ rule $\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + \ldots\ldots.. + y_{n-2}) + 4(y_1 + y_3 + y_5 + \ldots\ldots. + y_{n-1})\}$

where $h = \frac{x_n - x_0}{n}$, $n =$ number of intervals

Given $h = \frac{1}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$y = \frac{1}{1+x^2}$	1	0.9411	0.8	0.64	0.5

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{h}{3}\{(y_0 + y_4) + 2(y_2 + \ldots\ldots..) + 4(y_1 + y_3 + \ldots\ldots..)\}$$

https://doi.org/10.5281/zenodo.15289898

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{0.25}{3}\{(1+0.5)+2(0.8)+4(0.9411+0.64)\}$$

$$= \frac{0.25}{3}\{1.5+1.6+6.3244\}$$

$$= 0.7853$$

By actual Integration, we have

$$\int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_0^1$$

$$0.7853 = \tan^{-1} 1 - \tan^{-1} 0$$

$$0.7857 = \frac{\pi}{4}$$

$$3.1428 \approx \pi$$

6. Evaluate $\int_0^1 \frac{1}{1+x}\, dx$, using Simpson's $\frac{1}{3}$ rule with $h = 0.125$ and compare the values with exact value. **QC 20753 MA 6452 NOV 2018**

Given $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = \frac{1}{1+x}$	1	0.888	0.8	0.7272	0.666	0.6153	0.5714	0.533	0.5

By Simpson's $\frac{1}{3}$ rule, $\int_{x_0}^{x_n} y\, dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_6+\ldots\ldots+y_{n-2})+4(y_1+y_3+\ldots\ldots+y_{n-1})\}$

Actual Integration value

$$\int_0^1 \frac{1}{1+x}\, dx = \left[\log_e(1+x)\right]_0^1 = \log_e 2 - \log_e 1 = 0.6931$$

$$\int_0^1 \frac{1}{1+x} dx = \frac{h}{3}\{(y_0+y_8)+2(y_2+y_4+\ldots\ldots+y_8)+4(y_1+y_3+\ldots\ldots+y_7)\}$$

$$= \frac{0.125}{3}\{(1+0.5)+2(0.8+0.666+0.5714)+4(0.888+0.7272+0.6153+0.533)\}$$

$$= \frac{0.125}{3}\{1.5+4.0748+11.054\}$$

$$= 0.6928$$

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7. **Evaluate** $\int_{-3}^{3} x^4 \, dx$ **dividing the range of integration into 8 equal parts using Simpson's** $\frac{1}{3}$ **rule. Also compare the results with actual integration.** **QC 80610 MA 6452 NOV 2016**

Let $h = \frac{3-(-3)}{8} = 0.75$

x	-3	-2.25	-1.5	-0.75	0	0.75	1.5	2.25	3
$y = x^4$	81	25.63	5.06	0.316	0	0.316	5.06	25.63	81

By Simpson's $\frac{1}{3}$ rule, $\int_{-3}^{3} x^4 \, dx = \frac{h}{3}\left\{(y_0 + y_n) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1})\right\}$

$$\int_{-3}^{3} x^4 \, dx = \frac{h}{3}\left\{(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)\right\}$$

$$= \frac{0.75}{3}\left\{(81+81) + 2(5.06+5.06) + 4(25.63+0.316+0.316+25.63)\right\}$$

$$= \frac{0.75}{3}\left\{162 + 20.24 + 207.568\right\}$$

$$= 97.452$$

Actual Integration:

$$\int_{-3}^{3} x^4 \, dx = 2\int_{0}^{3} x^4 \, dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2}{5}(243) = 97.2$$

Comparing, the actual value of integration with the Simpson's $\frac{1}{3}$ integration values, we see that Simpson's rule give more accurate value.

8. **The velocity** v **of a particle at a distance** s **from a point on its linear path is given in the Following data:**

$s(m)$:	0	2.5	5	7.5	10	12.5	15	17.5	20
$v(m/s)$:	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 meters, using Simpson's $\frac{1}{3}$ **rule.** **QC 53252 MA 6459 MAY 2019**

We know that $v = \frac{ds}{dt}$, where $v =$ velocity, $s =$ distance, $t =$ time

Therefore $dt = \frac{1}{v} ds$ and hence $t = \int \frac{1}{v} \, ds$

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Given $h = 2.5$

s	0	2.5	5	7.5	10	12.5	15	17.5	20
v	16	19	21	22	20	17	13	11	9
$\frac{1}{v}$	1/16	1/19	1/21	1/22	1/20	1/17	1/13	1/11	1/9

By Simpson's $\frac{1}{3}$ rule, $t = \int_0^{20} \frac{1}{v}\ ds = \frac{h}{3}\left\{\left(\frac{1}{v_0}+\frac{1}{v_8}\right)+2\left(\frac{1}{v_2}+\frac{1}{v_4}+\frac{1}{v_6}\right)+4\left(\frac{1}{v_1}+\frac{1}{v_3}+\frac{1}{v_5}+\frac{1}{v_7}\right)\right\}$

$$s = \frac{2.5}{3}\left\{\left(\frac{1}{16}+\frac{1}{9}\right)+2\left(\frac{1}{21}+\frac{1}{20}+\frac{1}{13}\right)+4\left(\frac{1}{19}+\frac{1}{22}+\frac{1}{17}+\frac{1}{11}\right)\right\}$$

$$= \frac{2.5}{3}\{0.173611+0.34908+0.99127\}$$

$$= 1.2616\ mtrs$$

Numerical Integration – Simpson's 3/8 Rule

Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\ dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+.......+y_{n-2})+4(y_1+y_3+......+y_{n-1})\}$$

where $h = \frac{x_n - x_0}{n}$, n = number of intervals

It is obtained from the Newton Cote's formula by putting n =3 i.e. assuming that there are only four paired values of x & y or that the interpolating polynomial is a third degree polynomial.

Note:

It gives exact answer, if the integrand is a polynomial of degree 3

Number of intervals of the tabulated values of x, y must be multiple of 3.

Under what condition Simpson's $\frac{3}{8}$ rule can be applied and state the formula.
MA 6459 NOV 2015

1. **Evaluate $\int_{-3}^{3} x^4\ dx$ dividing the range of integration into 8 equal parts using Simpson's $\frac{3}{8}$ rule. Also compare the results with actual integration.** **QC 80610 MA 6452 NOV 2016**

Let $h = \frac{3-(-3)}{6} = 1$

x	-3	-2	-1	0	1	2	3
$y = x^4$	81	16	1	0	1	16	81

By Simpson's $\frac{3}{8}$ rule, $\int_{-3}^{3} x^4 \, dx = \frac{h}{3}\left\{(y_0 + y_n) + 2(y_2 + y_4 + ... + y_{n-2}) + 4(y_1 + y_3 + ... + y_{n-1})\right\}$

$$\int_{-3}^{3} x^4 \, dx = \frac{3h}{8}\left\{(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right\}$$

$$s = \frac{3\times 1}{8}\left\{(81+81) + 2(0) + 3(16+1+1+16)\right\}$$
$$= \frac{3}{8}\left\{162 + 0 + 102\right\}$$
$$= 99$$

Actual Integration:

$$\int_{-3}^{3} x^4 \, dx = 2\int_{0}^{3} x^4 \, dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2}{5}(243) = 97.2$$

Comparing, the actual value of integration with the Simpson's $\frac{3}{8}$ integration values, we see that the difference is significant.

Double Integration – Trapezoidal Rule

$$\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\, dxdy = \frac{hk}{4}\left[\begin{array}{c}(sum\ of\ \ f\ \ at\ corners) + 2(sum\ of\ \ remaining\ \ f\ \ at\ boundary\ ) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{array}\right]$$

1. Evaluate $\int_1^{1.4}\int_1^{1.4} \frac{1}{x+1} dxdy$ by Trapezoidal rule with $h = k = 0.1$. QC 50782 MA 6452 NOV 2017

Here $f(x,y) = \frac{1}{x+1}$ and $h = 0.1,\ k = 0.1.$

y \ x	1	1.1	1.2	1.3	1.4
1	0.5	0.476	0.455	0.435	0.416
1.1	0.5	0.476	0.455	0.435	0.416
1.2	0.5	0.476	0.455	0.435	0.416
1.3	0.5	0.476	0.455	0.435	0.416
1.4	0.5	0.476	0.455	0.435	0.416

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By Trapezoidal Rule

$$\int_1^{1.4}\int_1^{1.4} \frac{1}{x+1}\,dxdy = \frac{hk}{4}\left[\begin{array}{c}(sum\ of\ \ f\ \ at\ corners)+2(sum\ of\ \ remaining\ \ f\ \ at\ boundary\ ) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{array}\right]$$

$$=\frac{0.1\times 0.1}{4}\left[(1.832)+2(5.48)+4(4.098)\right]$$

$$=0.07296$$

2. Using Trapezoidal rule, evaluate $\int_1^2\int_1^2 \frac{1}{x+y}\,dxdy$ with $h=k=0.5$. QC 57506 MA 6452 MAY 2016

Here $f(x,y)=\frac{1}{x+y}$ and $h=0.5$, $k=0.5$.

x / y	1	1.5	2
1	0.5	0.4	0.333
1.5	0.4	0.333	0.286
2	0.333	0.286	0.25

By Trapezoidal Rule

$$\int_1^{2}\int_1^{2} \frac{1}{x+y}\,dxdy = \frac{hk}{4}\left[\begin{array}{c}(sum\ of\ \ f\ \ at\ corners)+2(sum\ of\ \ remaining\ \ f\ \ at\ boundary\ ) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{array}\right]$$

$$=\frac{0.5\times 0.5}{4}\left[(1.416)+2(1.372)+4(0.333)\right]$$

$$=0.3433$$

3. Using Trapezoidal rule evaluate $\int_0^1\int_0^1 \frac{1}{1+x+y}\,dxdy$ with $h=0.5$ along $x-$ direction and $k=0.25$ along $y-$ direction QC 27335 MA 6459 NOV 2015

Here $f(x,y)=\frac{1}{1+x+y}$ and $h=0.5$, $k=0.25$.

x / y	0	0.5	1
0	1	0.6666	0.5
0.25	0.8	0.5714	0.4444
0.5	0.6666	0.5	0.4
0.75	0.5714	0.4444	0.3636
1	0.5	0.4	0.3333

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By Trapezoidal Rule

$$\int_0^1 \int_0^1 \frac{1}{1+x+y}\, dxdy = \frac{hk}{4}\left[\begin{array}{c}(sum\ of\ \ f\ \ at\ corners)+2(sum\ of\ remaining\ \ f\ \ at\ boundary\ ) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{array}\right]$$

$$= \frac{0.5\times 0.25}{4}\left[(2.3333)+2(4.2746)+4(3.5538)\right]$$

$$= 0.7843$$

4. **Evaluate** $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\, dx\, dy$ **using trapezoidal rule.** **QC 53252 MA 6459 MAY 2019**

Here $f(x,y) = cos(x+y)$ and let $h = k = \frac{\pi}{4}$

y \ x	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
0	0	-0.7071	-1
$\frac{\pi}{4}$	-0.7071	-1	-0.7071
$\frac{\pi}{2}$	-1	-0.7071	0

By Trapezoidal Rule

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \cos(x+y)\, dx\, dy = \frac{hk}{4}\left[\begin{array}{c}(sum\ of\ \ f\ \ at\ corners)+2(sum\ of\ remaining\ \ f\ \ at\ boundary\ ) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{array}\right]$$

$$= \frac{\frac{\pi}{4}\times\frac{\pi}{4}}{4}\left[(-2)+2(-2.8284)+4(-1)\right]$$

$$= -1.7976$$

$$\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\,dxdy = \frac{hk}{9}\begin{bmatrix}(\textit{sum of } f \textit{ at corners}) \\ +2(\textit{sum of } f \textit{ at odd positions on the boundary}) \\ +4(\textit{sum of } f \textit{ at even positions on the boundary}) \\ +4(\textit{sum of } f \textit{ at odd positions on the odd row}) \\ +8(\textit{sum of } f \textit{ at even positions on the odd row}) \\ +8(\textit{sum of } f \textit{ at odd positions on the even row}) \\ +16(\textit{sum of } f \textit{ at even positions on the even row})\end{bmatrix}$$

5. **Evaluate $\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y}dxdy$ by Simpson's $\frac{1}{3}$ rule by taking $h=k=0.1$. QC 60045 MA 3251 APR 2022**

Let us evaluate the values of $f(x,y)=\frac{1}{x+y}$ at selected points. Given that $h=k=0.1$

y \ x	1	1.1	1.2	1.3	1.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4762	0.4545	0.4348	0.4167	0.4
1.2	0.4545	0.4348	0.4167	0.4	0.3846

By Simpson's $\frac{1}{3}$ rule, $\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y}dxdy$

$$= \frac{(0.1)(0.1)}{9}\begin{bmatrix}\begin{pmatrix}0.5+0.4167 \\ +0.4545+0.3846\end{pmatrix}+2(0.4545+0.4167)+4\begin{pmatrix}0.4762+0.4348+0.4 \\ +0.4762+0.4+0.4348\end{pmatrix} \\ +8(0.4348)+16(0.4545+0.4167)\end{bmatrix}$$

$$= \frac{(0.1)(0.1)}{9}[1.7558+1.7424+10.488+3.4784+13.9392]$$

$$=0.03489$$

6. **Using Simpson's $\frac{1}{3}$ rule, to evaluate $\int_0^1\int_0^1 \frac{1}{1+xy}dxdy$ with $\Delta x=\Delta y=0.25$.**

QC 41313 MA 6452 MAY 2018

Here $f(x,y)=\frac{1}{1+xy}$ and $h=0.25$, $k=0.25$.

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y \ x	0	0.25	0.5	0.75	1
0	1	1	1	1	1
0.25	1	0.941	0.888	0.842	0.8
0.5	1	0.888	0.8	0.727	0.666
0.75	1	0.842	0.727	0.64	0.571
1	1	0.8	0.666	0.571	0.5

By Simpson's Rule

$$\int_0^1\int_0^1 \frac{1}{1+xy}\,dxdy=\frac{hk}{9}\begin{bmatrix}(sum\ of\ \ f\ \ at\ corners)\\ +2(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ boundary)\\ +4(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ boundary)\\ +4(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ odd\ row)\\ +8(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ odd\ row)\\ +8(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ even\ row)\\ +16(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ even\ row)\end{bmatrix}$$

$$=\frac{0.25\times 0.25}{9}\begin{bmatrix}(3.5)+2(3.332)+4(6.742)+4(0.8)\\ +8(1.615)+8(1.615)+16(3.265)\end{bmatrix}$$

$$=0.8223$$

7. Evaluate $\int_0^2\int_1^2 \sin(9x+y)\,dxdy$ by Simpson's $\frac{1}{3}$ rule and Trapezoidal rule with $h=0.25$ and $k=0.5$.

QC 80610 MA 6452 NOV 2016

Here $f(x,y)=\sin(9x+y)$ and $h=0.25$, $k=0.5$.

y \ x	1	1.25	1.5	1.75	2
0	0.412	–0.967	0.803	–0.042	–0.75
0.5	–0.075	**–0.728**	**0.99**	**–0.516**	–0.342
1	–0.544	**–0.311**	**0.934**	**–0.863**	0.149
1.5	–0.879	**0.182**	**0.65**	**–0.999**	0.605
2	–0.99	0.632	0.206	–0.891	0.913

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By Trapezoidal Rule

$$\int_0^2 \int_1^2 \sin(9x+y)\, dxdy = \frac{hk}{4}\begin{bmatrix}(sum\ of\ \ f\ \ at\ corners)+2(sum\ of\ \ remaining\ \ f\ \ at\ boundary) \\ +4(sum\ of\ \ f\ \ at\ interiors)\end{bmatrix}$$

$$= \frac{0.25\times 0.5}{4}\left[(-0.415)+2(-1.345)+4(-0.661)\right]$$

$$= -0.17965$$

By Simpson's Rule

$$\int_0^2 \int_1^2 \sin(9x+y)\, dxdy = \frac{hk}{9}\begin{bmatrix}(sum\ of\ \ f\ \ at\ corners) \\ +2(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ boundary) \\ +4(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ boundary) \\ +4(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ odd\ \ row) \\ +8(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ odd\ \ row) \\ +8(sum\ of\ \ f\ \ at\ odd\ \ positions\ on\ the\ even\ row) \\ +16(sum\ of\ \ f\ \ at\ even\ \ positions\ on\ the\ even\ row)\end{bmatrix}$$

$$= \frac{0.25\times 0.5}{9}\begin{bmatrix}(-0.415)+2(0.614)+4(-1.959)+4(0.934) \\ +8(-1.174)+8(1.64)+16(-2.061)\end{bmatrix}$$

$$= -0.4518$$

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EXERCISE

Interpolation for Equal Intervals

1 Construct Newton's forward interpolation polynomial for the following data:

x	:	1	2	3	4	5
$f(x)$	:	1	–1	1	–1	1

and hence find $f(3.5)$, $f'(3.5)$. (QC 51579 MA 2266 MAY 2014)

2 Find $y(22)$, given that

x:	20	25	30	35	40	45
$y(x)$:	354	332	291	260	231	204

(QC 11491 MA 2266 NOV 2012)

3 From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46 and 63.

Age x	:	45	50	55	60	65
Premium y	:	114.84	96.16	83.32	74.48	68.48

(QC 11395 MA 2266 MAY 2011)

Interpolation for Unequal Intervals

1 Use the Newton divided difference formula to calculate $f(3)$, $f'(3)$ and $f''(3)$ from the following table:

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

(QC E3126 MA 2266 MAY 2010)

2 Using Newton's divided difference formula, find the value of $f(2)$ *and* $f(14)$ from the following table:

x	4	5	7	10	11	13
$f(x)$	48	100	294	90	1210	2028

(QC 31528 MA 2266 NOV 2013)

3 Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(4)$ for

x	:	1	3	5	7
$f(x)$	:	24	120	336	720

(QC 51579 MA 2266 MAY 2014)

4. Using Lagrange's method, find the value of $f(3)$ from the following table.

x	:	0	1	2	3
$f(x)$	:	2	3	12	147

(QC 31528 MA 2266 NOV 2013)

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Numerical Differentiation

1 From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1.2$

x:	1	1.2	1.4	1.6	1.8	2	2.2
y:	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

(QC E3126 MA 2266 MAY 2010)

2 The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year (x)	:	1931	1941	1951	1961	1971
Population In Thousands (y)	:	40.62	60.8	79.95	103.56	132.65

(QC 21528 MA 2266 MAY 2013)

3. Find $y'(1)$, if

x:	0	2	3	4	7	9
$y(x)$:	4	26	58	112	466	922

(QC 11491 MA 2266 NOV 2012)

Numerical Integration

1. A rocket is launched from the ground. Its acceleration registered during the first 80 seconds is given below. Using Trapezoidal rule and Simpson's $\frac{1}{3}$ rule, find the velocity of the rocket at $t = 80$ sec. (QC E3126 MA 2266 MAY 2010)

t(sec)	0	10	20	30	40	50	60	70	80	
$f(cm/\text{sec})$	30	31.73	33.34	35.47	37.75	40.33	43.25	46.69	40.67	

2. The velocity V of a particle at distances from a point on its path is given by the table

S feet :	0	10	20	30	40	50	60
V feet/s :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Trapezoidal and Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule. QC 72071 MA 6452 MAY 2017

3. The table below gives the velocity v of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at $t = 12$ seconds, using Simpson's rule.

t	0	2	4	6	8	10	12
v	4	6	16	34	60	94	136

(QC 11395 MA 2266 MAY 2011)

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4. Evaluate the length of the curve $3y = x^3$ from $(0,0)$ to $\left(1,\frac{1}{3}\right)$, using Simpson's $\frac{1}{3}$ rule using 8 sub-intervals. (Hint: Arc length $=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$) (QC E3126 MA 2266 MAY 2010)

5. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x \, dx$ by Trapezoidal and Simpson's rule. Verify your answer with integration. (QC 21528 MA 2266 MAY 2013)

Double Integration

1. Taking $h = k = \frac{\pi}{4}$, evaluate $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sqrt{\sin(x+y)}\, dx\, dy$ by Simpson's $\frac{1}{3}$ rule. (QC 31528 MA 2266 NOV 2013)

2. Evaluate $\int_1^2 \int_1^2 \frac{1}{x+y} dxdy$ by Simpson's and Trapezoidal rule with $h = 0.5,\ k = 0.25$.

QC 77194 MA 6452 APR 2015

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UNIT V – NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Taylor's series method

To find the solution of the first order differential equation $y' = f(x, y)$ subject to the initial condition $y(x_0) = y_0$.

Expand $y(x)$ about the point $x = x_0$ in Taylor's series. Then

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + ...$$

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_o'' + ...$$

$$y_1 = y(x_1) = y_0 + \frac{(x_1-x_0)}{1!}y_0' + \frac{(x_1-x_0)^2}{2!}y_o'' + ...$$

$$y_1 = y(x_1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_o'' + ... \text{ where } h = x_1 - x_0 \text{ and } y_o^{(r)} = \left[\frac{d^r y}{dx^r}\right]_{x=x_0}$$

Similarly, by expanding $y(x)$ about the point $x = x_1$ in Taylor's series, we can find y_2. Continuing in this way, we can find $y_3, y_4, ...$

Note:

- If h is small, Taylor's method is powerful.
- Successive derivatives of a function are known, Taylor's method is useful.
- If $f(x, y)$ is somewhat complicated and the calculation of higher order derivatives becomes tedious, then this method fails.
- If we retain terms up to h^n and neglecting h^{n+1} and higher powers, the Taylor's algorithm is said to be of order n. The truncation error is $O\left(h^{n+1}\right)$.

1. **In solving $y' = f(x, y)$, $y(x_0) = y_0$ by Taylor's method, if h^4 and higher powers of h are omitted, what is the order of this method.**

   Order is three

2. **Write the merits of the Taylor's method of solution.**

   This method gives a straight forward adoption to develop the solution as an infinite series. This single step method will be useful for finding the starting values for powerful methods like RK method, Milne's method etc.

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Solved Problems

1. **Solve** $\frac{dy}{dx} = x^2 - y,$ **given** $y(0) = 1$ **and find the values of** $y(0.1)$ **and** $y(0.2)$ **using Taylor series method, correct to four decimal places.** **QC 60045 MA 3251 APR 2022**

Given $y' = x^2 - y$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1)$$

$$= 1 - 0.1 + 0.005 + 0.0001$$

$$= 0.9051$$

$y' = x^2 - y$ i.e. $y_0' = x_0^2 - y_0 = 0 - 1 = -1$

$y'' = 2x - y'$ i.e. $y_0'' = 2x_0 - y_0' = 2(0) - (-1) = 1$

$y''' = 2 - y''$ i.e. $y_0''' = 2 - y_0'' = 2 - 1 = 1$

To find $y_2 = y(0.2)$ by Taylor's Method:

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

$$y_2 = y(0.2) = 0.905 + \frac{0.1}{1!}(-0.895) + \frac{(0.1)^2}{2!}(1.09) + \frac{(0.1)^3}{3!}(0.904)$$

$$= 0.905 - 0.08985 + 0.005 + 0.0054$$

$$= 0.8256$$

$y_1' = x_1^2 - y_1 = (0.1)^2 - 0.9051 = -0.8951$

$y_1'' = 2x_1 - y_1' = 2(0.1) - (-0.8951) = 1.0951$

$y_1''' = 2 - y_1'' = 2 - 1.0951 = 0.9049$

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2. **Compute $y(0.1)$ correct to 4 decimal places if $y(x)$ satisfies $y' = x + y,\ y(0) = 1$, by Taylor's series method.**

QC 72071 MA 6452 MAY 2017

Given $f(x, y) = x + y$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2)$$

$$= 1 + 0.1 + 0.01$$

$$= 1.11$$

$y' = x + y$ i.e. $y_0' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y'$ i.e. $y_0'' = 1 + y_0' = 1 + 1 = 2$

3. **Solve $y' = y^2 + x,\ y(0) = 1$ using Taylor's series method for $y(0.1)$ and $y(0.2)$.**

QC 41313 MA 6452 MAY 2018

Given $y' = y^2 + x$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8)$$

$$= 1 + 0.1 + 0.015 + 0.001$$

$$= 1.116$$

$y' = y^2 + x, \quad y'' = 2yy' + 1, \quad y''' = 2yy'' + 2y'y'$

$$y_0' = y_0^2 + x_0 = 1^2 + 0 = 1$$

$$y_0'' = 2y_0 y_0' + 1 = 2(1) + 1 = 3$$

$$y_0''' = 2y_0 y_0'' + 2y_0' y_0' = 6 + 2 = 8$$

https://doi.org/10.5281/zenodo.15289913

To find $y_2 = y(0.2)$ by Taylor's Method:

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

$$y_2 = y(0.2) = 1.116 + \frac{0.1}{1!}(1.345) + \frac{(0.1)^2}{2!}(3.53) + \frac{(0.1)^3}{3!}(11.5)$$

$$= 1.116 + 0.1345 + 0.0176 + 0.0019$$

$$= 1.2702$$

$$y_1' = y_1^2 + x_1 = 1.116^2 + 0.1 = 1.345$$

$$y_1'' = 2y_1 y_1' + 1 = 2(1.116)(1.134) + 1 = 3.53$$

$$y_1''' = 2y_1 y_1'' + 2y_1' y_1' = 2(1.116)(3.53) + 2(1.34)^2 = 11.5$$

4. Using Taylor's method, find y at $x = 0.1$ when $y' = x^2 - y,\ y(0) = 1$. QC 80610 MA 6452 NOV 2016

Given $f(x,y) = x^2 - y$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(1)$$

$$= 1 - 0.1 + 0.005$$

$$= 0.905$$

$$y' = x^2 - y$$

$$y_0' = x_0^2 - y_0 = 0 - 1 = -1$$

$$y'' = 2x - y'$$

$$y_0'' = 2x_0 - y_0' = 1$$

Euler's method

Algorithm for Euler method to solve $y' = f(x, y)$, given initial condition is $y(x_0) = y_0$

Let $x_i = x_0 + ih,\ i = 0,1,2,....$

The equation of tangent at $(x_0,\ y_0)$ to the curve is

$$y - y_0 = y'_{(x_0, y_0)}(x - x_0)$$

$$= f(x_0, y_0)(x - x_0)$$

$y = y_0 + f(x_0, y_0)(x - x_0)$, the value of y on the tangent corresponding to $x = x_1$

$\therefore$ the value of y on the curve = the value of y on the tangent corresponding to $x = x_1$

$$\therefore \quad y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

1. Write down the Euler formula for $y' = f(x, y),\ y(x_0) = y_0$. **QC 80610 MA 6452 NOV 2016**

Euler's formula to find $y_{m+1} = y_m + h\ f(x_m,\ y_m),\ \ m = 0, 1, 2, \ldots\ldots, n$

2. Use Euler's formula to find $y(0.2)$ and $y(0.4)$ given $y' = x + y,\ y(0) = 1$. QC 60045 MA 3251 APR 22

Given data can be tabulated as follows:

$$f(x, y) = x + y, \quad and \quad let \quad h = 0.2$$

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.2$	$y_1 = y(0.2)$
$x_2 = 0.4$	$y_2 = y(0.4)$

By Euler's formula,

$y_1 = y_0 + h\ f(x_0, y_0)$ $\qquad$ $y_2 = y_1 + h\ f(x_1, y_1)$

$= 1 + (0.2)[x_0 + y_0]$ $\qquad$ $= 1.2 + (0.2)[x_1 + y_1]$

$= 1 + (0.2)[0+1]$ $\qquad$ $= 1.2 + (0.2)[0.2+1.2]$

$= 1.2$ $\qquad$ $= 1.48$

3. If $y' = -y,\ y(0) = 1$ then find $y(0.1)$ by Euler method. **QC 27331 MA 6452 NOV 2015**

$$f(x, y) = -y, \quad and \quad given \quad h = 0.1$$

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1)$

By Euler's formula,

$$y_1 = y_0 + h\ f(x_0, y_0)$$

$$= 1 + (0.1)\,f(0,1)$$

$$= 1 + (0.1)[-1]$$

$$= 0.9$$

4. Given $y' = y$ and $y(0) = 1$ determine the values of y at $x = 0.01(0.01)0.04$ by Euler method.
QC 53250 MA 6452 MAY 2019

Given $f(x,y) = y$ and $h = 0.01$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.01$	$y_1 = y(0.01) = ?$
$x_2 = 0.02$	$y_2 = y(0.02) = ?$
$x_3 = 0.03$	$y_3 = y(0.03) = ?$
$x_4 = 0.04$	$y_4 = y(0.04) = ?$

To find $y_1 = y(0.01)$ using Euler's method. To find $y_2 = y(0.02)$ using Euler's method.

$$y_1 = y_0 + h\, f\,(x_0, y_0)$$
$$= 1 + (0.01)\, f\,(0, 1)$$
$$= 1 + (0.01)(1)$$
$$= 1.01$$

$$y_2 = y_1 + h\, f\,(x_1, y_1)$$
$$= 1.01 + (0.01)\, f\,(0.01, 1.01)$$
$$= 1.01 + (0.01)(1.01)$$
$$= 1.0201$$

To find $y_3 = y(0.03)$ using Euler's method.

$$y_3 = y_2 + h\, f\,(x_2, y_2)$$
$$= 1.0201 + (0.01)\, f\,(0.02, 1.0201)$$
$$= 1.0201 + (0.01)(1.0201)$$
$$= 1.0303$$

To find $y_4 = y(0.04)$ using Euler's method.

$$y_4 = y_3 + h\, f\,(x_3, y_3)$$
$$= 1.0303 + (0.01)\, f\,(0.03, 1.0303)$$
$$= 1.0303 + (0.01)(1.0303)$$
$$= 1.0406$$

5. Using Euler's method, find y of $x = 0.1$ if $\frac{dy}{dx} = 1 + xy,\ y(0) = 2$. QC 41313 MA 6452 MAY 2018

Given $f(x, y) = 1 + xy$ and $h = 0.1$	
x	y
$x_0 = 0$ $x_1 = 0.1$	$y_0 = y(0) = 2$ $y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ using Euler's method.

$$y_1 = y_0 + h\, f\,(x_0, y_0)$$
$$= 2 + (0.1)\, f\,(0, 2)$$
$$= 2 + (0.1)(1 + (0)(2))$$
$$= 2.1$$

6. Find $y(0.01)$ by using Euler's method, given that $\frac{dy}{dx} = -y,\ y(0) = 1$. QC 20753 MA 6452 NOV 2018

Given $f(x, y) = -y$ and $h = 0.01$	
x	y
$x_0 = 0$ $x_1 = 0.01$	$y_0 = 1$ $y_1 = y(0.01) = ?$

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To find $y_1 = y(0.01)$ by Euler Method

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.01 f(0,1)$$
$$= 1 + 0.01[-1]$$
$$= 0.99$$

7. **Find $y(0.1)$ by Euler's method, if $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0.1$** **QC 57506 MA 6452 MAY 2016**

Given $f(x,y) = x^2 + y^2$ and $h = 0.1$	
x	y
$x_0 = 0$ $x_1 = 0.1$	$y_0 = 0.1$ $y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ by Euler Method

$$y_1 = y_0 + h\ f(x_0, y_0)$$
$$= 0.1 + 0.1 f(0, 0.1)$$
$$= 0.1 + 0.1\left[0^2 + (0.1)^2\right]$$
$$= 0.101$$

Modified Euler's method

1. **State Modified Euler Formula.** **QC 53250 MA 6452 MAY 2019**

Modified Euler formula : $y_{n+1} = y_n + h\, f\left[x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right],\ n = 0,1,2,\ldots,n-1$

2. **Given $\frac{dy}{dx} = y - x^2 + 1,\ y(0) = 0.5.$ Find $y(0.2)$ by Modified Euler method**

QC 41313 MA 6452 MAY 2018

Given $f(x,y) = y - x^2 + 1$ and $h = 0.1$	
x	y

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$x_0 = 0$	$y_0 = 0.5$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

By Modified Euler method ,

$$y_1 = y_0 + h\, f\left[x_0 + \frac{h}{2}\ ,\ y_0 + \frac{h}{2}\, f\,(\,x_0\,,\,y_0\,)\right]$$

$$= 0.5\ + (0.1)\ f\left[0 + \frac{0.1}{2}\ ,\ 0.5 + \frac{0.1}{2}\, f\,(0,\,0.5)\right]$$

$$= 0.5 + (0.1)\ f\left[\ 0.05\,,\,0.5 + (0.05\,)(0.5 - 0^2\ +1)\right]$$

$$= 0.5 + (0.1)\ f\left(\ 0.05,\,0.075\ \right)$$

$$= 0.5 + (0.1)\,[0.075 - (0.05)^2 + 1]$$

$$= 0.6073$$

$$y_2 = y_1 + h\, f\left[x_1 + \frac{h}{2}\ ,\ y_1 + \frac{h}{2}\, f\,(x_1,\,y_1)\right]$$

$$= 0.6073\ + (0.1)\ f\left[0.1 + \frac{0.1}{2},\ 0.6073 + \frac{0.1}{2}\ f\ (0.1,\,0.6073)\right]$$

$$= 0.6073 + (0.1)\ f\left[\ 0.15\,,\ \ 0.6073 + (0.05\,)\,(0.6073 - 0.1^2 + 1)\right]$$

$$= 0.6073 + (0.1)\ f\left(\,0.15,\,0.6872\,\right)$$

$$= 0.6073 + (0.1)\,[\ 0.6872 - (0.15)^2 + 1]$$

$$= 0.7737$$

3. Solve $\frac{dy}{dx} = x + y$, $y(0) = 1$ by Modified Euler's method to find $y(0.2)$ with $h = 0.1$.

QC 20753 MA 6452 NOV 2018

Given $f(x,y) = (x+y)$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

To find $y_1 = y(0.1)$ using Modified Euler's method.

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$$y_1 = y_0 + h\, f\left[x_0 + \frac{h}{2}\ ,\ y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$= 1 + (0.1)\, f\left[0 + \frac{0.1}{2}\ ,\ 1 + \frac{0.1}{2} f(0, 1)\right]$$

$$= 1 + (0.1)\, f\left[\,0.05\, ,\, 1 + (0.05)\,(0+1)\right]$$

$$= 1 + (0.1)\, f\left(\,0.05, 1.05\,\right)$$

$$= 1 + (0.1)\,[\,0.05 + 1.05]$$

$$= 1.11$$

To find $y_2 = y(0.2)$ using Modified Euler's method.

$$y_2 = y_1 + h\, f\left[x_1 + \frac{h}{2}\ ,\ y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 1.21 + (0.1)\, f\left[0.1 + \frac{0.1}{2}\ ,\ 1.21 + \frac{0.1}{2} f(0.1, 1.21)\right]$$

$$= 1.21 + (0.1)\, f\left[\,0.15\, ,\, 1.21 + (0.05)\,(0.1 + 1.21)\right]$$

$$= 1.21 + (0.1)\, f\left(\,0.15, 1.2755\,\right)$$

$$= 1.21 + (0.1)\,[\,0.15 + 1.2755]$$

$$= 1.3675$$

4. Use Euler's modified method to find $y(0.1)$ and $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1.$

QC 60045 MA 3251 APR 2022

Given $f(x,y) = x^2 + y^2$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

By Modified Euler method ,

$$y_1 = y_0 + h\ f\left[x_0 + \frac{h}{2}\ ,\ y_0 + \frac{h}{2}\ f\ (x_0\ ,\ y_0\ )\right]$$

$$= 1 + (\ 0.1\ )\ f\ \left[0 + \frac{0.1}{2}\ ,\ 1 + \frac{0.1}{2}\ f\ (\ 0, 1\ )\right]$$

$$= 1 + (\ 0.1)\ f\left[\ 0.05\ , 1 + (\ 0.05\ )\ (\ 0^2 + 1^2\ )\right]$$

$$= 1 + (0.1)\ f\ (\ 0.05, 1.05\ )$$

$$= 1 + (\ 0.1\ )\ [\ (0.05)^2 + (1.05)^2\ ]$$

$$= 1 + 0.1105$$

$$y(0.1) = 1.1105$$

$$y_2 = y_1 + h\ f\left[x_1 + \frac{h}{2}\ ,\ y_1 + \frac{h}{2}\ f\ (x_1\ ,\ y_1\ )\right]$$

$$= 1.1105 + (0.1)\ f\left[0.1 + \frac{0.1}{2}\ ,\ 1.1105 + \frac{0.1}{2}\ f\ (0.1, 1.1105\ )\right]$$

$$= 1.1105 + (\ 0.1)\ f\left[\ 0.15\ ,\ 1.1105 + (\ 0.05\ )\ (\ 0.1^2 + 1.1105^2\ )\right]$$

$$= 1.1105 + (0.1)\ f\ (\ 0.15, 1.1727\ )$$

$$= 1.1105 + (\ 0.1\ )\ [\ (0.15)^2 + (1.1727)^2\ ]$$

$$= 1 + 0.13977$$

$$y(0.2) = 1.13977$$

Runge Kutta method of order IV

1. Write the formula for RK method of second order .

$y_1 = y_0 + \Delta y$ where $\Delta y = k_2$ and $k_1 = hf(x_0, y_0)$ and $k_2 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

Note: RK method of order II is Modified Euler Method

2. Write the formula for RK method of second order .

$y_1 = y_0 + \Delta y$ where $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$

$k_1 = hf(x_0, y_0)$, $k_2 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$ and $k_3 = h\ f\left(x_0 + h, y_0 + 2k_2 - k_1\right)$

Note: RK method of order II is Modified Euler Method

3. Define fourth order R.K method. **QC 53250 MA 6452 MAY 2019**

$$k_1 = h \times f\left(x_n, y_n\right)$$

$$k_2 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h \times f\left(x_n + h, y_n + k_3\right)$$

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$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \text{ where } n = 0,1,2,...,n-1$$

Note:

i. Error of RK method is $k.h^5$.

ii. If $f(x,y) = f(x)$, then in the RK method of order IV, Δy reduces to Simpson's $\frac{1}{3}$ rule.

4. In the derivation of 4th order RK formula, why it is called 4th order.

We know that the truncation error in RK method of order n is $k.h^{n+1}$, k is constant.

Since the truncation error is $k.h^5$, we name the method as RK method of order IV.

In other words, the slopes at four points are used to construct the weighted average slope m.

5. Write down the formula for classical Runge-Kutta method. **QC 20817 MA 8452 APR 2022**

$$k_1 = h \times f(x_n, y_n)$$

$$k_2 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h \times f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \text{ where } n = 0,1,2,...,n-1$$

Solved Problems

1. Using Runge-Kutta method of order IV, solve $y' = y + x,\ y(0) = 1$ to find $y(0.1)$.

QC 41313 MA 6452 MAY 2018

Given $f(x,y) = (x+y)$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ using RK method.

$$k_1 = h \times f(x_0, y_0) = (0.1) \times f(0,1) = (0.1) \times [0+1] = 0.1$$

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$$k_2 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right) = (0.1)\times f(0.05, 1.05) = (0.1)\times[0.05+1.05] = 0.11$$

$$k_3 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{0.11}{2}\right) = (0.1)\times f(0.05, 1.055) = (0.1)\times[0.05+1.055] = 0.1105$$

$$k_4 = h\times f(x_0+h, y_0+k_3) = (0.1)\times f(0+0.1, 1+0.1105) = (0.1)\times f(0.1, 1.1105) = (0.1)\times[0.1+1.1105] = 0.12105$$

$$y_1 = y(0.1) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 1+\frac{1}{6}[0.1+2(0.11)+2(0.1105)+(0.12105)]$$

$$= 1.1103$$

2. Using R-K method of fourth order, find the value y of at $x=0.1$, if y satisfies the equation $\frac{dy}{dx} = y^2 + x$ given that $y=1$ when $x=0$, correct to 3 decimal places. **QC 50782 MA 6452 NOV 2017**

Given $f(x,y) = x+y^2$ and $h=0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ using RK method.

$$k_1 = h\ f\ (x_0, y_0)$$
$$= (0.1)\ f(0,1)$$
$$= (0.1)\left[0+1^2\right]$$
$$= 0.1$$

$$k_2 = h\ f\ \left(x_0+\frac{h}{2},\ y_0+\frac{k_1}{2}\right)$$
$$= (0.1)\ f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right)$$
$$= (0.1)\ f(0.05, 1.05)$$
$$= (0.1)\left[0.05+1.05^2\right]$$
$$= 0.1153$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)\ f\left(0 + \frac{0.1}{2},\, 1 + \frac{0.1153}{2}\right)$$

$$= (0.1)\ f\left(0.05,\, 1.0576\right)$$

$$= (0.1)\left[0.05 + 1.0576^2\right]$$

$$= 0.1168$$

$$k_4 = h\, f\left(x_0 + h,\, y_0 + k_3\right)$$

$$= (0.1)\ f\left(0 + 0.1,\, 1 + 0.1168\right)$$

$$= (0.1)\ f\left(0.1,\, 1.1168\right)$$

$$= (0.1)\left[0.1 + 1.1168^2\right]$$

$$= 0.1347$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}\left[0.1 + 2(0.1153) + 2(0.1168) + 0.1347\right] = 0.1165$$

Now $y_1 = y_0 + \Delta y = 1 + 0.1165 = 1.1165$

3. By fourth order Runge-Kutta method find $y(0.2)$ from $\frac{dy}{dx} = y - x$, $y(0) = 2$ taking $h = 0.1$

QC 72071 MA 6452 MAY 2017

Given $f(x,y) = y - x$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 2$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

To find $y_1 = y(0.1)$ using RK method.

$$k_1 = h \times f\left(x_0, y_0\right) = (0.1) \times f\left(0,\, 2\right) = (0.1) \times \left[2 - 0\right] = 0.2$$

$$k_2 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) \times f\left(0 + \frac{0.1}{2},\, 2 + \frac{0.2}{2}\right) = (0.1) \times f\left(0.05,\, 2.1\right) = (0.1) \times \left[2.1 - 0.05\right] = 0.205$$

$$k_3 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) \times f\left(0 + \frac{0.1}{2},\, 2 + \frac{0.205}{2}\right) = (0.1) \times f\left(0.05,\, 2.103\right) = (0.1) \times \left[2.103 - 0.05\right] = 0.205$$

$$k_4 = h \times f\left(x_0 + h, y_0 + k_3\right) = (0.1) \times f\left(0 + 0.1,\, 2 + 0.205\right) = (0.1) \times f\left(0.1,\, 2.205\right) = (0.1) \times \left[2.205 - 0.1\right] = 0.2105$$

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$$y_1 = y(0.1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 2 + \frac{1}{6}[0.2 + 2(0.205) + 2(0.205) + (0.2105)]$$

$$= 2.205$$

To find $y_2 = y(0.2)$ using RK method.

$$k_1 = h \times f(x_1, y_1) = (0.1) \times f(0.1,\ 2.205) = (0.1) \times [2.205 - 0.1] = 0.2105$$

$$k_2 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1) \times f\left(0.1 + \frac{0.1}{2},\ 2.205 + \frac{0.2105}{2}\right)$$

$$= (0.1) \times f(0.15,\ 2.31) = (0.1) \times [2.31 - 0.15] = 0.216$$

$$k_3 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1) \times f\left(0.1 + \frac{0.1}{2},\ 2.205 + \frac{0.216}{2}\right)$$

$$= (0.1) \times f(0.15,\ 2.313) = (0.1) \times [2.313 - 0.15] = 0.216$$

$$k_4 = h \times f(x_1 + h,\ y_1 + k_3) = (0.1) \times f(0.1 + 0.1,\ 2.205 + 0.216)$$

$$= (0.1) \times f(0.2,\ 2.421) = (0.1) \times [2.421 - 0.2] = 0.222$$

$$y_2 = y(0.2) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 2.205 + \frac{1}{6}[0.2105 + 2(0.216) + 2(0.216) + 0.222] = 2.421$$

4. Given $y' = y - x^2,\ y(0.6) = 1.7379$ find $y(0.7),\ y(0.8)$ using R-K method of fourth order.

QC 80610 MA 6452 NOV 2016

Given $f(x,y) = y - x^2$ and $h = 0.1$	
x	y
$x_0 = 0.6$	$y_0 = 1.7379$
$x_1 = 0.7$	$y_1 = y(0.7) = ?$
$x_2 = 0.8$	$y_2 = y(0.8) = ?$

By RK method ,

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$$k_1 = h\,f\,(x_0, y_0)$$
$$= (0.1)\ f(0.6, 1.7379)$$
$$= (0.2)\left[1.7379 - 0.6^2\right]$$
$$= 0.2755$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$= (0.1)\ f\left(0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.2755}{2}\right)$$
$$= (0.1)\ f\left(0.65, 1.87565\right)$$
$$= (0.1)\left[1.87565 - 0.65^2\right]$$
$$= 0.1453$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$
$$= (0.1)\ f\left(0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.1453}{2}\right)$$
$$= (0.1)\ f\left(0.65, 1.811\right)$$
$$= (0.1)\left[1.811 - 0.65^2\right]$$
$$= 0.1388$$

$$k_4 = h\,f\left(x_0 + h,\ y_0 + k_3\right)$$
$$= (0.1)\ f\left(0.6 + 0.1, 1.7379 + 0.1388\right)$$
$$= (0.1)\ f\left(0.7, 1.8767\right)$$
$$= (0.1)\left[1.8767 - 0.7^2\right]$$
$$= 0.1386$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}\left[0.2755 + 2(0.1453) + 2(0.1388) + 0.1386\right] = 0.1637$$

Now $y_1 = y_0 + \Delta y = 1.7379 + 0.1637 = 1.9016$

Again by RK method ,

$$k_1 = h\,f\,(x_1, y_1)$$
$$= (0.1)\ f(0.7, 1.9016)$$
$$= (0.1)\left[1.9016 - 0.7^2\right]$$
$$= 0.1412$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$
$$= (0.1)\ f\left(0.7 + \frac{0.1}{2}, 1.9016 + \frac{0.1412}{2}\right)$$
$$= (0.1)\ f\left(0.75, 1.972\right)$$
$$= (0.1)\left[1.972 - 0.75^2\right]$$
$$= 0.1409$$

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$$k_3 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$$= (0.1)\ f\left(0.7 + \frac{0.1}{2},\ 1.9016 + \frac{0.1409}{2}\right)$$

$$= (0.1)\ f(0.75,\ 1.9721)$$

$$= (0.1)\left[1.9721 - 0.75^2\right]$$

$$= 0.1409$$

$$k_4 = h\, f\left(x_1 + h,\ y_1 + k_3\right)$$

$$= (0.1)\ f(0.7 + 0.1,\ 1.9016 + 0.1409)$$

$$= (0.1)\ f(0.8,\ 2.0425)$$

$$= (0.1)\left[2.0425 - 0.8^2\right]$$

$$= 0.1402$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\left[0.1412 + 2(0.1409) + 2(0.1409) + 0.1402\right]$$

$$= 0.3526$$

Now $y_2 = y_1 + \Delta y = 1.9016 + 0.3526 = 2.2542$

5. Compute $y(0.1)$ given $\frac{dy}{dx} + y + xy^2 = 0,\ y(0) = 1,$ by taking $h = 0.1$ using Runge-Kutta method of order 4, correct to 4 decimal accuracy. **QC 60045 MA 3251 APR 2022**

Given $f(x,y) = -y - xy^2$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$

To find $y_1 = y(0.1)$ by RK method of order IV:

$$k_1 = h \times f(x_0, y_0) = (0.1) \times f(0,1) = (0.1) \times \left[-1 - 0(1)^2\right] = -0.1$$

https://doi.org/10.5281/zenodo.15289913

$$k_2 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{-0.1}{2}\right)$$

$$= (0.1)\times f\left(0.05,\ 0.95\right) = (0.1)\times\left[-0.95-(0.05)(0.95)^2\right] = -0.0995$$

$$k_3 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{-0.0995}{2}\right)$$

$$= (0.1)\times f\left(0.05,\ 0.95025\right) = (0.1)\times\left[-0.95025-(0.05)(0.95025)^2\right] = -0.0995$$

$$k_4 = h\times f\left(x_0+h, y_0+k_3\right) = (0.1)\times f\left(0+0.1,\ 1-0.0995\right)$$

$$= (0.1)\times f\left(0.1,\ 0.9005\right) = (0.1)\times\left[-0.9005-(0.1)(0.9005)^2\right] = -0.0982$$

$$y_1 = y(0.1) = y_0 + \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1+\frac{1}{6}\left[-0.1+2(-0.0995)+2(-0.0995)+(-0.0982)\right]$$

$$= 0.9006$$

6. Apply fourth order R-K method to find $y(0.2)$, given $y' = x+y,\ \ y(0)=1$. QC 53250 MA 6452 MAY 2019

Given $f(x,y) = (x+y)$ and let $h=0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$

To find $y_1 = y(0.1)$ using RK method.

$$k_1 = h\times f\left(x_0, y_0\right) = (0.1)\times f\left(0, 1\right) = (0.1)\times\left[0+1\right] = 0.1$$

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$$k_2 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right)$$

$$= (0.1)\times f\left(0.05, 1.05\right) = (0.1)\times\left[0.05+1.05\right] = 0.11$$

$$k_3 = h\times f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0.1)\times f\left(0+\frac{0.1}{2}, 1+\frac{0.11}{2}\right)$$

$$= (0.1)\times f\left(0.05, 1.055\right) = (0.1)\times\left[0.05+1.055\right] = 0.1105$$

$$k_4 = h\times f\left(x_0+h, y_0+k_3\right) = (0.1)\times f\left(0+0.1, 1+0.1105\right)$$

$$= (0.1)\times f\left(0.1, 1.1105\right) = (0.1)\times\left[0.1+1.1105\right] = 0.12105$$

$$y_1 = y(0.1) = y_0 + \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1+\frac{1}{6}\left[0.1+2(0.11)+2(0.1105)+(0.12105)\right]$$

$$= 1.1103$$

To find $y_2 = y(0.2)$ using RK method.

$$k_1 = h\times f\left(x_1, y_1\right) = (0.1)\times f\left(0.1, 1.1103\right)$$

$$= (0.1)\times\left[0.1 + 1.1103\right] = 0.121$$

$$k_2 = h\times f\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = (0.1)\times f\left(0.1+\frac{0.1}{2}, 1.1103+\frac{0.121}{2}\right)$$

$$= (0.1)\times f\left(0.15, 1.1708\right) = (0.1)\times\left[0.15+1.1708\right] = 0.1321$$

$$k_3 = h\times f\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = (0.1)\times f\left(0.1+\frac{0.1}{2}, 1.1103+\frac{0.1321}{2}\right)$$

$$= (0.1)\times f\left(0.15, 1.1763\right) = (0.1)\times\left[0.15+1.1763\right] = 0.1326$$

$$k_4 = h\times f\left(x_1+h, y_1+k_3\right) = (0.1)\times f\left(0.1+0.1, 1.1103+0.1326\right)$$

$$= (0.1)\times f\left(0.2, 1.2429\right) = (0.1)\times\left[0.2+1.2429\right] = 0.1443$$

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$$y_2 = y(0.2) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1103 + \frac{1}{6}[0.121 + 2(0.1321) + 2(0.1326) + (0.1443)]$$

$$= 1.2427$$

Milne's predictor corrector methods

1. What is predictor –corrector methods.

   In these methods we use a pair of formulae to solve the differential equation $y' = f(x, y),\ y(x_0) = y_0$

   Using the predictor formula, we predict a value of y at y_k say.

   Then we use corrector formula to correct the value of y_k and get a better approximation for y .

2. What is meant by single step methods?

   The methods which use only the information from the last step computed are called single step methods.

3. Why Milne's method of solving ODE is called multi step method.

   In addition to the last step computed values, Milne's method uses additional information such as the function value and the value of derivatives. $\therefore$ it is called multi step method.

4. What do we mean by saying that a method is self starting? Not self starting?

   Methods used to solve differential equation does not use more than the initial value of y is called self starting. The methods which requires the value of y at prior points is called not self starting.

5. State Milne's formula.

   $$y_{n+1,\,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n],$$

   $$y_{n+1,\,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}],$$ where $n = 3, 4, 5, \ldots$

   Note: Predictor – Corrector methods are not self starting method

6. How many initial values are required to predict the next value by Milne's method.

   Four values are required

Solved Problems

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1. **Use Milne's predictor-corrector formula to find $y(0.4)$, given** **QC 60045 MA 3251 APR 2022**

$\frac{dy}{dx} = 0.5\left[1+x^2\right]y^2,\ y(0)=1,\ y(0.1)=1.06,\ y(0.2)=1.12\ and\ y(0.3)=1.21.$

Given $y' = 0.5\left[1+x^2\right]y^2$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = 1.06$
$x_2 = 0.2$	$y_2 = y(0.2) = 1.12$
$x_3 = 0.3$	$y_3 = y(0.3) = 1.21$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$
$$= 1 + \frac{4(0.1)}{3}\left[2(0.5674) - (0.6523) + 2(0.7979)\right]$$
$$= 1.2771$$

$$y_1' = 0.5\left(1+x_1^2\right)y_1^2$$
$$= (0.5)\left[1+0.1^2\right]1.06^2 = 0.5674$$
$$y_2' = 0.5\left(1+x_2^2\right)y_2^2$$
$$= (0.5)\left[1+0.2^2\right]1.12^2 = 0.6523$$
$$y_3' = 0.5\left(1+x_3^2\right)y_3^2$$
$$= (0.5)\left[1+0.3^2\right]1.21^2 = 0.7979$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$
$$= 1.12 + \frac{0.1}{3}\left[0.6523 + 4(0.7979) + 0.9459\right]$$
$$= 1.2796$$

$$y_4' = 0.5\left(1+x_4^2\right)y_4^2$$
$$= (0.5)\left[1+0.4^2\right]1.2771^2$$
$$= 0.9459$$

2. **Given** $y' = \frac{1}{2}xy$ **with** $y(0)=1,\ y(0.1)=1.01,\ y(0.2)=1.022,\ y(0.3)=1.023.$**. Find** $y(0.4)$ **using Milne's method.** **QC 20817 MA 8452 APR 2022**

Given $y' = \frac{1}{2}xy$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = 1.01$
$x_2 = 0.2$	$y_2 = y(0.2) = 1.022$
$x_3 = 0.3$	$y_3 = y(0.3) = 1.023$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0505) - (1.022) + 2(0.15345)\right]$$

$$= 0.9181$$

$$y_1' = 0.5\left(x_1 y_1\right)$$
$$= (0.5)\left[(0.1)(1.01)\right] = 0.0505$$
$$y_2' = 0.5\left(x_2 y_2\right)$$
$$= (0.5)\left[(0.2)(1.022)\right] = 0.1022$$
$$y_3' = 0.5\left(x_3 y_3\right)$$
$$= (0.5)\left[(0.3)(1.023)\right] = 0.15345$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.022 + \frac{0.1}{3}\left[0.1022 + 4(0.15345) + 0.1836\right]$$

$$= 1.0519$$

$$y_4' = 0.5\left(x_4 y_4\right)$$
$$= (0.5)\left[(0.4)(0.9181)\right]$$
$$= 0.1836$$

3. **Given** $5x\frac{dy}{dx} + y^2 - 2 = 0, \quad y(4) = 1, \ y(4.1) = 1.0049, \quad y(4.2) = 1.0097, \quad y(4.3) = 1.0143.$ **Compute** $y(4.4)$ **by Milne's Predictor Corrector method.** **QC 41313 MA 6452 MAY 2018**

Given $y' = \frac{2 - y^2}{5x}$ and $h = 0.1$	
x	y

https://doi.org/10.5281/zenodo.15289913

$x_0 = 4$	$y_0 = 1$
$x_1 = 4.1$	$y_1 = y(0.1) = 1.0049$
$x_2 = 4.2$	$y_2 = y(0.2) = 1.0097$
$x_3 = 4.3$	$y_3 = y(0.3) = 1.0143$
$x_4 = 4.4$	$y_4 = y(0.4) = ?$

By Milne's Predictor formula

$$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - 1.0049^2}{5(4.1)} = 0.0483$$

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0483) - (0.0466) + 2(0.0452)\right]$$

$$= 1.01872$$

$$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2 - 1.0097^2}{5(4.2)} = 0.0466$$

$$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2 - 1.0143^2}{5(4.3)} = 0.0452$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0466 + 4(0.0452) + 0.0437\right]$$

$$= 1.01869$$

$$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2 - 1.01872^2}{5(4.4)} = 0.0437$$

4. **Given** $\frac{dy}{dx} = x^3 + y$ **,** $y(0) = 2,\ y(0.2) = 2.073,\ y(0.4) = 2.452,\ y(0.6) = 3.023$ **compute** $y(0.8)$ **by Milne's method.** **QC 50782 MA 6452 NOV 2017**

Given $y' = x^3 + y$ and $h = 0.2$	
x	y

https://doi.org/10.5281/zenodo.15289913

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = y(0.2) = 2.073$
$x_2 = 0.4$	$y_2 = y(0.4) = 2.452$
$x_3 = 0.6$	$y_3 = y(0.6) = 3.023$
$x_4 = 0.8$	$y_4 = y(0.8) = ?$

To find $y_4 = y(0.8)$ by Milne's Method:

Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 2 + \frac{4(0.1)}{3}[2(2.081) - (2.516) + 2(3.239)]$$

$$= 3.083$$

$$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$$

$$y_2' = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$$

$$y_3' = x_3^3 + y_3 = (0.6)^3 + 3.023 = 3.239$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 2.452 + \frac{0.1}{3}[2.516 + 4(3.239) + 3.595]$$

$$= 3.087$$

$$y_4' = x_4^3 + y_4 = (0.8)^3 + 3.083 = 3.595$$

Adam's – Bashforth predictor corrector method

1. **Give an example of multi step method.** **QC 20817 MA 8452 APR 2022**

Milne's Predictor Corrector Method and Adam's Bashforth Predictor Corrector Method

2. **Write down the Adam-Bashforth predictor and corrector formulae.** **QC 60045 MA 3251 APR 22**

$$y_{n+1,\,P} = y_n + \frac{h}{24}[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'],$$

$$y_{n+1,\,C} = y_n + \frac{h}{24}[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'],$$ where n = 3, 4, 5, ...

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3. **What is the main difference between single and multistep methods in solving first order ordinary differential equation?** **QC 50782 MA 6452 NOV 2017**

Single step method	Multi step method
To find y value at current point, one value is required at prior point	To find y value at current point, minimum three values are required at prior points
Self starting method	Not self starting method
No explicit method for estimating truncation error	Truncation error can be obtained easily
To improve the accuracy step size must be very small and the process is tedious.	By repeated applications of corrector formula, accuracy can be improved easily

4. **What are single step and multistep methods? Give an example.** **QC 27331 MA 6452 NOV 2015**

A method which requires one preceding value (say y_m) to find y_{m+1} is called single step method.
Example : Taylor's series method, Euler's method.

A method which requires more than one preceding value, other than initial value, to find y_{m+1} is called multi step method.

Example : Milne's Predictor-Corrector method, Adam's Predictor-Corrector method.

Solved Problems

1. **Given** $y' = \frac{1}{2}xy$ **with** $y(0)=1,\ y(0.1)=1.01,\ y(0.2)=1.022,\ y(0.3)=1.023.$. **Find** $y(0.4)$ **using Adam-Bashforth method** **QC 20817 MA 8452 APR 2022**

Given $y' = 0.5(xy)$ and $h=0.1$	
x	y

$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = 1.01$
$x_2 = 0.2$	$y_2 = y(0.2) = 1.022$
$x_3 = 0.3$	$y_3 = y(0.3) = 1.023$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

By Adams Predictor formula

$$y_{4,\,P} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$
$$= 1.023 + \frac{0.1}{24}\left[55(0.1534) - 59(0.1022) + 37(0.0505) - 9(0)\right]$$
$$= 1.0408$$

$$y_0' = 0.5\left(x_0 y_0\right) = (0.5)\left[(0)(1)\right] = 0$$
$$y_1' = 0.5\left(x_1 y_1\right) = (0.5)\left[(0.1)(1.01)\right]$$
$$= 0.0505$$
$$y_2' = 0.5\left(x_2 y_2\right) = (0.5)\left[(0.2)(1.022)\right]$$
$$= 0.1022$$
$$y_3' = 0.5\left(x_3 y_3\right) = (0.5)\left[(0.3)(1.023)\right]$$
$$= 0.1534$$

By Adams Corrector formula

$$y_{4,\,C} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$
$$= 1.023 + \frac{0.1}{24}\left[9(0.2082) + 19(0.15345) - 5(0.1022) + 0.0505\right]$$
$$= 1.041$$

$$y_4' = 0.5\left(x_4 y_4\right)$$
$$= (0.5)\left[(0.4)(1.0408)\right]$$
$$= 0.2082$$

Combined Methods

1. Name any two self starting method.

   (i) Euler Method (ii) Taylor's Method

2. What is the main difference between Euler's and Modified Euler,s method.

   RK method of order I is known as Euler Method

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RK method of order II is known as Modified Euler Method

3. What is the relation between Euler's and Taylors method.

   If we omit h^2 and higher powers of h from the Taylor series expansion, we get

   Euler method.

4. State the main difference between Taylors method over RK method.

   In Taylor series method, there is a chance to check the values computed earlier, whereas computing errors cannot be easily determined in RK method.

5. Why RK method is better than Taylors method.

   Higher order derivatives are not required

   Convergence is fast

   Very less labour work is involved

6. Why RK method is better than Predictor – Corrector methods

   (i) RK method is self starting

   (ii) RK method permits an easy change in the step size

7. What are the disadvantages of RK method over Milne's Method

   RK method is time consuming whereas PC methods are more efficient

   RK method provides no easily obtainable information about the truncation error

Solved Problems

1. Given $\frac{dy}{dx} = xy + y^2$, $y(0)=1,\ y(0.1)=1.1169,\ y(0.2)=1.2773,$ find (i) $y(0.3)$ by RK method of IV order (ii) $y(0.4)$ by Milne's method. **QC 20753 MA 6452 NOV 2018**

Given $y' = xy + y^2$ and $h = 0.1$	
x	y

$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = 1.1169$
$x_2 = 0.2$	$y_2 = y(0.2) = 1.2773$
$x_3 = 0.3$	$y_3 = y(0.3) = ?$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

To find $y_3 = y(0.3)$ by RK method of order IV:

$$k_1 = h\times f\left(x_2, y_2\right) = (0.1)\times f\left(0.2,\ 1.2773\right) = (0.1)\times\left[(0.2)(1.2773) + (1.2773)^2\right] = 0.1886$$

$$k_2 = h\times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.1)\times f\left(0.2 + \frac{0.1}{2}, 1.2773 + \frac{0.1886}{2}\right)$$

$$= (0.1)\times f\left(0.25,\ 1.3716\right) = (0.1)\times\left[(0.25)(1.371) + (1.371)^2\right] = 0.2224$$

$$k_3 = h\times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1)\times f\left(0.2 + \frac{0.1}{2}, 1.2773 + \frac{0.2224}{2}\right)$$

$$= (0.1)\times f\left(0.25,\ 1.388\right) = (0.1)\times\left[(0.25)(1.388) + (1.388)^2\right] = 0.2274$$

$$k_4 = h\times f\left(x_2 + h,\ y_2 + k_3\right) = (0.1)\times f\left(0.2 + 0.1, 1.2773 + 0.2274\right)$$

$$= (0.1)\times f\left(0.3,\ 1.5047\right) = (0.1)\times\left[(0.3)(1.5047) + (1.5047)^2\right] = 0.2715$$

$$y_3 = y(0.3) = y_2 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1.2773 + \frac{1}{6}\left[0.1886 + 2(0.2224) + 2(0.2274) + (0.2715)\right]$$

$$= 1.5039$$

To find $y_4 = y(0.4)$ by Milne's Method:

Predictor formula

https://doi.org/10.5281/zenodo.15289913

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$
$$= 1 + \frac{4(0.1)}{3}\left[2(1.359) - (1.8869) + 2(2.7128)\right]$$
$$= 1.834$$

$$y_1' = x_1 y_1 + y_1^2 = (0.1)(1.1169) + 1.1169^2 = 1.359$$
$$y_2' = x_2 y_2 + y_2^2 = (0.2)(1.2773) + 1.2773^2 = 1.8869$$
$$y_3' = x_3 y_3 + y_3^2 = (0.3)(1.5039) + 1.5039^2 = 2.7128$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$
$$= 1.2773 + \frac{0.1}{3}\left[1.8869 + 4(2.7128) + 4.097\right]$$
$$= 1.8384$$

$$y_4' = x_4 y_4 + y_4^2$$
$$= (0.4)(1.834) + (1.834)^2 = 4.097$$

2. **Using Taylor's series method, solve** $\frac{dy}{dx} = xy + y^2,\ y(0) = 1$ **at** $x = 0.1,\ 0.2,\ 0.3$**. Continue the solution at** $x = 0.4$ **by Milne's predictor-corrector method. QC 72071 MA 6452 MAY 2017**

Given $y' = xy + y^2$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$
$x_3 = 0.3$	$y_3 = y(0.3) = ?$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots$$

$$y' = xy + y^2 \qquad y'' = xy' + y + 2yy'$$
$$y''' = (x + 2y)y'' + 2y'^2 + 2y'$$

https://doi.org/10.5281/zenodo.15289913

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(10)$$

$$= 1 + 0.1 + 0.015 + 0.001$$

$$= 1.116$$

$$y_0' = x_0 y_0 + y_0^2 = (0)(1) + 1^2 = 1$$

$$y_0'' = y_0 + x_0 y_0' + 2 y_0 y_0' = 1 + (0)(1) + 2(1) = 3$$

$$y_0''' = \left(x_0 + 2y_0\right) y_0'' + 2y_0'^2 + 2y_0' = (2)(3) + 2 + 2 = 10$$

To find $y_2 = y(0.2)$ by Taylor's Method:

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

$$y_2 = y(0.2)$$

$$= 1.116 + \frac{0.1}{1!}(1.357) + \frac{(0.1)^2}{2!}(4.28) + \frac{(0.1)^3}{3!}(16.32)$$

$$= 1.116 + 0.1357 + 0.0214 + 0.0027$$

$$= 1.2758$$

$$y_1' = x_1 y_1 + y_1^2 = (0.1)(1.116) + 1.116^2 = 1.357$$

$$y_1'' = y_1 + x_1 y_1' + 2 y_1 y_1' = 1.116 + (0.1)(1.357) + 2(1.514)$$

$$= 4.28$$

$$y_1''' = \left(x_1 + 2y_1\right) y_1'' + 2y_1'^2 + 2y_1'$$

$$= 9.84 + 3.68 + 2.71 = 16.32$$

To find $y_3 = y(0.3)$ by Taylor's Method:

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \ldots\ldots$$

$$y_3 = y(0.3)$$

$$= 1.2758 + \frac{0.1}{1!}(1.875) + \frac{(0.1)^2}{2!}(6.43) + \frac{(0.1)^3}{3!}(28.4)$$

$$= 1.2758 + 0.1875 + 0.032 + 0.0047$$

$$= 1.4899$$

$$y_2' = x_2 y_2 + y_2^2 = (0.2)(1.273) + 1.273^2 = 1.875$$

$$y_2'' = y_2 + x_2 y_2' + 2 y_2 y_2' = 1.273 + (0.2)(1.875) + 2(2.39)$$

$$= 6.432$$

$$y_2''' = \left(x_2 + 2y_2\right) y_2'' + 2y_2'^2 + 2y_2'$$

$$= 17.69 + 7.02 + 3.75 = 28.46$$

To find $y_4 = y(0.4)$ by Milne's Method:

Predictor formula

https://doi.org/10.5281/zenodo.15289913

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4(0.1)}{3}[2(1.357) - (1.875) + 2(2.6667)] \qquad y_3' = x_3y_3 + y_3^2 = (0.3)(1.4899) + 1.4899^2 = 2.6667$$

$$= 1.8229$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.2773 + \frac{0.1}{3}[1.875 + 4(2.6667) + 4.052]$$

$$y_4' = x_4y_4 + y_4^2 = (0.4)(1.8229) + (1.8229)^2 = 4.052$$

$$= 1.8304$$

3. **If** $\frac{dy}{dx} = y^2 + x^2$, $y(0) = 1$, **find** $y(0.1)$, $y(0.2)$ *and* $y(0.3)$ **by Taylor's series method. Hence find** $y(0.4)$ **by Milne's predictor-corrector method.** **QC 57506 MA 6452 MAY 2016**

Given $f(x,y) = y^2 + x^2$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$
$x_3 = 0.3$	$y_3 = y(0.3) = ?$
$x_4 = 0.4$	$y_4 = y(0.4) = ?$

To find $y_1 = y(0.1)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8)$$

$$= 1 + 0.1 + 0.01 + 0.00133$$

$$= 1.1113$$

$y' = x^2 + y^2$ i.e. $y_0' = x_0^2 + y_0^2 = 0 + 1 = 1$

$y'' = 2x + 2yy'$ i.e.

$y_0'' = 2x_0 + 2y_0 y_0' = 2(0) + 2(1)(1) = 2$

$y''' = 2 + 2yy'' + 2y'^2$ i.e.

$y_0''' = 2 + 2y_0 y_0'' + 2y_0'^2 = 2 + 2.1.2 + 2.1^2 = 8$

To find $y_2 = y(0.2)$ by Taylor's Method:

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots\ldots$$

$$y_2 = y(0.2) = 1.1113 + \frac{0.1}{1!}(1.244) + \frac{(0.1)^2}{2!}(2.965) + \frac{(0.1)^3}{3!}(10.12)$$

$$= 1.1113 + 0.1244 + 0.0148 + 0.00168$$

$$= 1.252$$

$y' = x^2 + y^2$ i.e.

$y_1' = x_1^2 + y_1^2 = 0.01 + 1.234 = 1.244$

$y'' = 2x + 2yy'$ i.e.

$y_1'' = 2x_1 + 2y_1 y_1' = 2(0.1) + 2(1.1113)(1.244)$

$= 2.965$

$y''' = 2 + 2yy'' + 2y'^2$ i.e.

$y_1''' = 2 + 2y_1 y_1'' + 2y_1'^2$

$= 2 + 2.(1.11)(2.96) + 1.24^2 = 10.12$

To find $y_3 = y(0.3)$ by Taylor's Method:

$$y_3 = y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \ldots\ldots$$

$$y_3 = y(0.3)$$

$$= 1.252 + \frac{0.1}{1!}(1.608) + \frac{(0.1)^2}{2!}(4.426) + \frac{(0.1)^3}{3!}(18.254)$$

$$= 1.252 + 0.1608 + 0.0221 + 0.00304$$

$$= 1.4379$$

$y' = x^2 + y^2$ i.e.

$y_2' = x_2^2 + y_2^2 = 0.04 + 1.568 = 1.608$

$y'' = 2x + 2yy'$ i.e.

$y_2'' = 2x_2 + 2y_2 y_2' = 2(0.2) + 2(1.252)(1.608)$

$= 4.426$

$y''' = 2 + 2yy'' + 2y'^2$ i.e.

$y_2''' = 2 + 2y_2 y_2'' + 2y_2'^2$

$= 2 + 2(1.252)(4.426) + (2)1.608^2 = 18.254$

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To find $y_4 = y(0.4)$ by Milne's Predictor-Corrector Method:

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.244) - (1.608) + 2(2.1575)\right]$$

$$= 1.6926$$

$$y_1' = x_1^2 + y_1^2 = 0.01 + 1.234 = 1.244$$

$$y_2' = x_2^2 + y_2^2 = 0.04 + 1.568 = 1.608$$

$$y_3' = x_3^2 + y_3^2 = 0.09 + 2.067 = 2.1575$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.252 + \frac{0.1}{3}\left[1.608 + 4(2.1575) + 3.024\right]$$

$$y_4' = x_4^2 + y_4^2 = 0.16 + 2.864 = 3.024$$

$$= 1.694$$

4. **If** $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$**,** $y(0) = 1$**, find** $y(0.2)$, $y(0.4)$, $f(0.6)$ **by Runge-Kutta method. Hence find** $y(0.8)$ **by Milne's method.** **QC 57506 MA 6452 MAY 2016**

Given $f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$ and $h = 0.2$	
x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.2$	$y_1 = y(0.2) = ?$
$x_2 = 0.4$	$y_2 = y(0.4) = ?$
$x_3 = 0.6$	$y_3 = y(0.6) = ?$
$x_4 = 0.8$	$y_4 = y(0.8) = ?$

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By RK method ,

$$k_1 = h\ f\ (x_0, y_0)$$
$$= (0.2)\ f(0,1)$$
$$= (0.2)\left[\frac{1^2 - 0^2}{1^2 + 0^2}\right]$$
$$= 0.2$$

$$k_2 = h\ f\ \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$= (0.2)\ f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right)$$
$$= (0.2)\ f\left(0.1,\ 1.1\right)$$
$$= (0.2)\left[\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right]$$
$$= 0.1967$$

$$k_3 = h\ f\ \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$
$$= (0.2)\ f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.1967}{2}\right)$$
$$= (0.2)\ f\left(0.1,\ 1.09836\right)$$
$$= (0.2)\left[\frac{1.09836^2 - 0.1^2}{1.09836^2 + 0.1^2}\right]$$
$$= 0.1967$$

$$k_4 = h\ f\ \left(x_0 + h,\ y_0 + k_3\right)$$
$$= (0.2)\ f\left(0 + 0.2,\ 1 + 0.1967\right)$$
$$= (0.2)\ f\left(0.02,\ 1.1967\right)$$
$$= (0.2)\left[\frac{1.1967^2 - 0.02^2}{1.1967^2 + 0.02^2}\right]$$
$$= 0.1891$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}\left[0.2 + 2(0.1967) + 2(0.1967) + 0.1891\right] = 0.19598$$

Now $y_1 = y_0 + \Delta y = 1 + 0.19598 = 1.19598$

Again by RK method ,

$$k_1 = h\ f\ (x_1, y_1)$$
$$= (0.2)\ f(0.2,\ 1.196)$$
$$= (0.2)\left[\frac{1.196^2 - 0.2^2}{1.196^2 + 0.2^2}\right]$$
$$= 0.1891$$

$$k_2 = h\ f\ \left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$
$$= (0.2)\ f\left(0.2 + \frac{0.2}{2},\ 1.196 + \frac{0.1891}{2}\right)$$
$$= (0.2)\ f\left(0.3,\ 1.291\right)$$
$$= (0.2)\left[\frac{1.291^2 - 0.3^2}{1.291^2 + 0.3^2}\right]$$
$$= 0.1795$$

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$$k_3 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$$= (0.2)\ f\left(0.2 + \frac{0.2}{2},\ 1.196 + \frac{0.1795}{2}\right)$$

$$= (0.2)\ f\left(0.3,\ 1.2858\right)$$

$$= (0.2)\left[\frac{1.2858^2 - 0.3^2}{1.2858^2 + 0.3^2}\right]$$

$$= 0.1793$$

$$k_4 = h\, f\left(x_1 + h,\ y_1 + k_3\right)$$

$$= (0.2)\ f\left(0.2 + 0.2,\ 1.196 + 0.1793\right)$$

$$= (0.2)\ f\left(0.4,\ 1.3753\right)$$

$$= (0.2)\left[\frac{1.3753^2 - 0.4^2}{1.3753^2 + 0.4^2}\right]$$

$$= 0.1688$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\left[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688\right]$$

$$= 0.17925$$

Now $y_2 = y_1 + \Delta y = 1.196 + 0.17925 = 1.3753$

Again by RK method ,

$$k_1 = h\, f\,(x_2, y_2)$$

$$= (0.2)\ f(0.4,\ 1.375)$$

$$= (0.2)\left[\frac{1.375^2 - 0.4^2}{1.375^2 + 0.4^2}\right]$$

$$= 0.1687$$

$$k_2 = h\, f\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_2}{2}\right)$$

$$= (0.2)\ f\left(0.4 + \frac{0.2}{2},\ 1.375 + \frac{0.1687}{2}\right)$$

$$= (0.2)\ f\left(0.5,\ 1.459\right)$$

$$= (0.2)\left[\frac{1.459^2 - 0.5^2}{1.459^2 + 0.5^2}\right]$$

$$= 0.1579$$

$$k_3 = h\, f\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_2}{2}\right)$$

$$= (0.2)\ f\left(0.4 + \frac{0.2}{2},\ 1.375 + \frac{0.1579}{2}\right)$$

$$= (0.2)\ f\left(0.5,\ 1.4539\right)$$

$$= (0.2)\left[\frac{1.453^2\ - 0.5^2}{1.453^2\ + 0.5^2}\right]$$

$$= 0.1576$$

$$k_4 = h\, f\ \left(x_2 + h,\ y_2 + k_3\right)$$

$$= (0.2)\ f\left(0.4 + 0.2,\ 1.375 + 0.1576\right)$$

$$= (0.2)\ f\left(0.6,\ 1.5326\right)$$

$$= (0.2)\left[\frac{1.532^2\ - 0.6^2}{1.532^2\ + 0.6^2}\right]$$

$$= 0.1468$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\left[0.1687 + 2(0.1579) + 2(0.1576) + 0.1468\right]$$

$$= 0.1577$$

Now $y_3 = y_2 + \Delta y = 1.3753 + 0.1722 = 1.5475$

To find $y_4 = y(0.8)$ by Milne's Predictor-Corrector Method:

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.945) - (0.844) + 2(0.7384)\right]$$

$$= 1.3364$$

$$y_1' = \frac{y_1^2 - x_1^2}{y_1^2 + x_1^2} = \left[\frac{1.195^2 - 0.2^2}{1.195^2 + 0.2^2}\right] = 0.945$$

$$y_2' = \frac{y_2^2 - x_2^2}{y_2^2 + x_2^2} = \left[\frac{1.3753^2 - 0.4^2}{1.3753^2 + 0.4^2}\right] = 0.844$$

$$y_3' = \frac{y_3^2 - x_3^2}{y_3^2 + x_3^2} = \left[\frac{1.547^2 - 0.6^2}{1.547^2 + 0.6^2}\right] = 0.7384$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.252 + \frac{0.1}{3}\left[0.844 + 4(0.7384) + 0.4744\right]$$

$$= 1.3944$$

$$y_4' = \frac{y_4^2 - x_4^2}{y_4^2 + x_4^2} = \left[\frac{1.3364^2 - 0.8^2}{1.3364^2 + 0.8^2}\right] = 0.4724$$

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5. **Given $y' = x + y$ with $y(0) = 1$. Assume $h = 0.1$. Find $y(0.1)$ using Euler's method. Hence, find $y(0.2)$ using Modified Euler method. Hence, find $y(0.3)$ using Runge-Kutta method of fourth order.** **QC 20817 MA 8452 APR 2022**

Given $f(x, y) = x + y$ and $h = 0.1$	
x	y
$x_0 = 0$	$y_0 = y(0) = 1$
$x_1 = 0.1$	$y_1 = y(0.1) = ?$
$x_2 = 0.2$	$y_2 = y(0.2) = ?$
$x_3 = 0.3$	$y_3 = y(0.3) = ?$

To find $y_1 = y(0.1)$ using Euler's method.

$$y_1 = y_0 + h\, f\,(x_0, y_0)$$

$$= 1 + (0.1)(x_0 + y_0)$$

$$= 1 + (0.1)(0 + 1)$$

$$= 1.1$$

To find $y_2 = y(0.2)$ using Modified Euler's method.

$$y_2 = y_1 + h\, f\left[x_1 + \frac{h}{2}\ ,\ y_1 + \frac{h}{2}\, f\,(x_1, y_1)\right]$$

$$= 1.1 + (0.1)\ f\ \left[0.1 + \frac{0.1}{2}\ ,\ 1.1 + \frac{0.1}{2}\, f(0.1, 1.1)\right]$$

$$= 1.1 + (0.1)\ f\left[\ 0.15\ , 1.1 + (0.05)\ (0.1 + 1.1)\right]$$

$$= 1.1 + (0.1)\ f\ (\ 0.15, 1.16\ )$$

$$= 1.1 + (0.1)\,[\,0.15 + 1.16\,]$$

$$= 1.231$$

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To find $y_3 = y(0.3)$ by RK method of order IV:

$$k_1 = h\times f\left(x_2, y_2\right) = (0.1)\times f\left(0.2,\, 1.231\right) = (0.1)\times\left[0.2 + 1.231\right] = 0.1431$$

$$k_2 = h\times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.1)\times f\left(0.2 + \frac{0.1}{2}, 1.231 + \frac{0.1431}{2}\right)$$

$$= (0.1)\times f\left(0.25,\, 1.30255\right) = (0.1)\times\left[0.25 + 1.30255\right] = 0.1553$$

$$k_3 = h\times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1)\times f\left(0.2 + \frac{0.1}{2}, 1.231 + \frac{0.1553}{2}\right)$$

$$= (0.1)\times f\left(0.25,\, 1.3086\right) = (0.1)\times\left[0.25 + 1.3086\right] = 0.1558$$

$$k_4 = h\times f\left(x_2 + h, y_2 + k_3\right) = (0.1)\times f\left(0.2 + 0.1,\, 1.231 + 0.1558\right)$$

$$= (0.1)\times f\left(0.3,\, 1.3868\right) = (0.1)\times\left[0.3 + 1.3868\right] = 0.1686$$

$$y_3 = y(0.3) = y_2 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1.231 + \frac{1}{6}\left[0.1431 + 2(0.1553) + 2(0.1558) + (0.1686)\right]$$

$$= 1.3866$$

6. **Compute** $y(0.5)$**,** $y(1)$ *and* $y(1.5)$ **using Taylor's series for** $y' = \dfrac{x+y}{2}$ **with** $y(0) = 2$ **and hence find** $y(2)$ **using Milne's method.** **QC 27331 MA 6452 NOV 2015**

Given $f(x,y) = 0.5(x+y)$ and $h = 0.5$	
x	y
$x_0 = 0$	$y_0 = 2$
$x_1 = 0.5$	$y_1 = y(0.5) = ?$
$x_2 = 1$	$y_2 = y(1) = ?$
$x_3 = 1.5$	$y_3 = y(1.5) = ?$
$x_4 = 2$	$y_4 = y(2) = ?$

https://doi.org/10.5281/zenodo.15289913

To find $y_1 = y(0.5)$ by Taylor's Method:

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots$$

$$y_1 = y(0.5) = 2 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0.5)$$

$$= 2 + 0.1 + 0.005 + 0.00001$$

$$= 2.105$$

$y' = 0.5(x+y)$ i.e.

$$y_0' = 0.5(x_0 + y_0) = 0.5(0+2) = 1$$

$y'' = 0.5(1+y')$ i.e. $y_0'' = 0.5(1+y_0') = 0.5(1+1) = 1$

$y''' = 0.5y''$ i.e. $y_0''' = 0.5y_0'' = 0.5(1) = 0.5$

To find $y_2 = y(1)$ by Taylor's Method:

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

$$y_2 = y(1) = 2.105 + \frac{0.1}{1!}(1.303) + \frac{(0.1)^2}{2!}(1.152) + \frac{(0.1)^3}{3!}(0.575)$$

$$= 2.105 + 0.1303 + 0.00576 + 0.00009$$

$$= 2.241$$

$$y_1' = 0.5(x_1 + y_1) = 0.5(0.5 + 2.105) = 1.303$$

$$y_1'' = 0.5(1 + y_1') = 0.5(1 + 1.303) = 1.152$$

$$y_1''' = 0.5y_1'' = 0.5(1.152) = 0.575$$

To find $y_3 = y(1.5)$ by Taylor's Method:

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \ldots\ldots$$

$$y_3 = y(0.3)$$

$$= 2.241 + \frac{0.1}{1!}(1.621) + \frac{(0.1)^2}{2!}(1.311) + \frac{(0.1)^3}{3!}(0.655)$$

$$= 2.241 + 0.1621 + 0.0065 + 0.0001$$

$$= 2.4097$$

$$y_2' = 0.5(x_2 + y_2) = 0.5(1 + 2.241) = 1.621$$

$$y_2'' = 0.5(1 + y_2') = 0.5(1 + 1.621) = 1.311$$

$$y_2''' = 0.5y_2'' = 0.5(1.311) = 0.655$$

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To find $y_4 = y(2)$ by Milne's Predictor-Corrector Method:

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4(0.5)}{3}\left[2(1.303) - (1.621) + 2(1.955)\right] \qquad y_3' = 0.5\left(x_3 + y_3\right) = 0.5\left(1.5 + 2.4097\right) = 1.955$$

$$= 5.263$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 2.241 + \frac{0.5}{3}\left[1.621 + 4(1.955) + 3.632\right] \qquad y_4' = 0.5\left(x_4 + y_4\right) = 0.5\left(2 + 5.263\right) = 3.632$$

$$= 4.4198$$

7. Consider the IVP . $y' = 1 - y,\ \ y(0) = 0$ Using the Euler's method find $y(0.2)$ and Modified Euler method find $y(0.4)$ and $y(0.6)$ then by using Milne's method obtain $y(0.8)$.

QC 80610 MA 6452 NOV 2016

Given $f(x, y) = 1 - y$ and $h = 0.2$	
x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.2$	$y_1 = y(0.2) = ?$
$x_2 = 0.4$	$y_2 = y(0.4) = ?$
$x_3 = 0.6$	$y_3 = y(0.6) = ?$
$x_4 = 0.8$	$y_4 = y(0.8) = ?$

To find $y_1 = y(0.2)$ by Euler Method

https://doi.org/10.5281/zenodo.15289913

$$y_1 = y_0 + hf\left(x_0, y_0\right)$$

$$= 0 + 0.1 f\left(0, 0\right)$$

$$= 0.1\left[1 - 0\right]$$

$$= 0.1$$

To find $y_2 = y(0.4)$ and $y_3 = y(0.6)$ by Modified Euler Method

$$y_2 = y_1 + h f\left[x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 0.1 + (0.2) f\left[0.2 + \frac{0.2}{2},\ 0.1 + \frac{0.2}{2} f(0.2, 0.1)\right]$$

$$= 0.1 + (0.2)\ f\left[\ 0.3,\ 0.1 + (0.1)(1 - 0.1)\right]$$

$$= 0.1 + (0.2)\ f\left(0.3, 0.19\right)$$

$$= 0.1 + (0.2)[1 - 0.19]$$

$$= 0.262$$

$$y_3 = y_2 + h f\left[x_2 + \frac{h}{2},\ y_2 + \frac{h}{2} f(x_2, y_2)\right]$$

$$= 0.262 + (0.2)\ f\left[0.4 + \frac{0.2}{2},\ 0.262 + \frac{0.2}{2} f(0.4, 0.262)\right]$$

$$= 0.262 + (0.2)\ f\left[\ 0.5,\ 0.262 + (0.1)(1 - 0.262)\right]$$

$$= 0.262 + (0.2)\ f\left(0.5, 0.3358\right)$$

$$= 0.262 + (0.2)[1 - 0.3358]$$

$$= 0.3948$$

By Milne's Predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.9) - (0.738) + 2(0.3948)\right]$$

$$= 0.4937$$

$$y_1' = 1 - y_1 = 1 - 0.1 = 0.9$$

$$y_2' = 1 - y_2 = 1 - 0.262 = 0.738$$

$$y_3' = 1 - y_3 = 1 - 0.3948 = 0.6052$$

By Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 0.262 + \frac{0.2}{3}\left[0.738 + 4(0.6058) + 0.5062\right]$$

$$= 0.5065$$

$$y_4' = 1 - y_4 = 1 - 0.4937 = 0.5062$$

Solution of ODE - Finite Difference Method

Working Rule:

Consider the boundary value problem $y''(x)+a(x)y'(x)+b(x)y(x)=c(x)$ together with the boundary conditions $y(x_0)=y_0,\ y(x_n)=y_n$ where $x\in\left(x_0,x_n\right)$.

Replace $y'(x),\ y''(x)$ by the respective central difference formula

Substitute given boundary conditions and the indices values which gives a system of n equations in n variables.

Solving, we get the values of $y_1,\ y_2,.....,\ y_{n-1}$ at $x=x_1,\ x_2,.....,\ x_{n-1}$.

1. **Write down the finite difference scheme for solving** $y''+x+y=0,\ y(0)=y(1)=0.$
QC 41313 MA 6452 MAY 2018

The finite difference approximation is $\dfrac{y_{i-1}+y_{i+1}-2y_i}{h^2}=-x_i-y_i$

2. **Give the central difference approximations for** $y'(x),\ y''(x)$**.** **QC 57506 MA 6452 MAY 2016**

The central difference scheme for

$$y'(x)=\frac{y_{i+1}-y_{i-1}}{2h}$$

$$y''(x)=\frac{y_{i+1}+y_{i-1}-2y_i}{h^2},\ i=1,\ 2,\ ....,(n-1)$$

3. **Write the finite difference approximation for the equation** $\dfrac{d^2y}{dx^2}=x+y$ **QC 20753 MA 6452 NOV 2018**

The finite difference approximation is $\dfrac{y_{i-1}+y_{i+1}-2y_i}{h^2}=x_i+y_i$

Solved Problems

4. Solve the equation $\dfrac{d^2y}{dx^2}=x+y$ with boundary conditions $y(0)=1=y(1)$ by finite difference method, by taking 4 subintervals. QC 50782 MA 6452 NOV 2017

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Let $h=\frac{1-0}{4}=0.25, \quad n=4$	
x	y
$x_0=0$	$y_0=1$
$x_1=0.25$	$y_1=y(0.25)=?$
$x_2=0.5$	$y_2=y(0.5)=?$
$x_3=0.75$	$y_3=y(0.75)=?$
$x_4=1$	$y_4=y(1)=1$

The finite difference scheme for the given differential equation is

$$\frac{y_{i+1}+y_{i-1}-2y_i}{h^2}=x_i+y_i,\ i=1,\ 2,\ 3$$

$$16\left(y_{i+1}+y_{i-1}-2y_i\right)-y_i=x_i$$

$$16y_{i+1}-33y_i+16y_{i-1}=x_i$$

$$16y_2-33y_1+16y_0=x_1$$

$$16y_3-33y_2+16y_1=x_2$$

$$16y_4-33y_3+16y_2=x_3$$

Substituting the initial conditions, we have

$$16y_2-33y_1+16=\frac{1}{4} \qquad 16y_2-33y_1=-\frac{63}{4}.............(1)$$

$$16y_3-33y_2+16y_1=\frac{1}{2} \quad \text{i.e.} \quad 16y_3-33y_2+16y_1=\frac{1}{2}......(2)$$

$$16-33y_3+16y_2=\frac{3}{4} \qquad -33y_3+16y_2=-\frac{61}{4}..........(3)$$

(1) – (3) gives

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$$-33y_1 + 33y_3 = -\frac{63}{4} + \frac{61}{4}a$$

$$-33y_1 + 33y_3 = -\frac{1}{2}$$

$$y_1 - y_3 = \frac{1}{66} \text{........}(4)$$

$$y_3 = y_1 - \frac{1}{66} \text{.......}(5)$$

$$\textit{From } (1) \quad y_2 = \frac{1}{16}\left[-\frac{63}{4} + 33y_1\right] = -\frac{63}{64} + \frac{33}{16}y_1 \text{............}(6)$$

Substitute (5), (6) in (2)

$$16y_1 - 33\left[-\frac{63}{64} + \frac{33}{16}y_1\right] + 16\left[y_1 - \frac{1}{66}\right] = \frac{1}{2}$$

$$16y_1 + \frac{2112}{64} - \frac{1089}{16}y_1 + 16y_1 - \frac{16}{66} = \frac{1}{2}$$

$$-36.0625y_1 = -32.2575$$

$$y_1 = 0.8944$$

From (5), $y_3 = y_1 - \frac{1}{66} = 0.08944 - \frac{1}{66} = 0.8793$

From (6), $y_2 = -\frac{63}{64} + \frac{33}{16}y_1 = -\frac{63}{64} + \frac{33}{16}(0.08944) = 0.8603$

5 Solve the differential equation $\frac{d^2y}{dx^2} - y = x$ with $y(0) = 0,\ y(1) = 0$ and $h = 0.25$ by finite difference method. **QC 72071 MA 6452 MAY 2017**

Given $h = 0.25$, $n = 4$	
x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.25$	$y_1 = y(0.25) = ?$
$x_2 = 0.5$	$y_2 = y(0.5) = ?$
$x_3 = 0.75$	$y_3 = y(0.75) = ?$
$x_4 = 1$	$y_4 = y(1) = 0$

https://doi.org/10.5281/zenodo.15289913

Given differential equation is $\frac{d^2y}{dx^2} = x + y$

The finite difference scheme for the given differential equation is

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = x_i + y_i,\ i = 1,\ 2,\ 3$$

$$16\left(y_{i+1} + y_{i-1} - 2y_i\right) - y_i = x_i$$

$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i$$

$$16y_2 - 33y_1 + 16y_0 = x_1$$

$$16y_3 - 33y_2 + 16y_1 = x_2$$

$$16y_4 - 33y_3 + 16y_2 = x_3$$

Substituting the initial conditions, we have

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

i.e.

$$-132y_1 + 64y_2 = 1 \ldots\ldots\ldots\ldots(1)$$

$$32y_1 - 66y_2 + 32y_3 = 1 \ldots\ldots(2)$$

$$64y_2 - 132y_3 = 3 \ldots\ldots\ldots\ldots\ldots(3)$$

(1) – (3) gives

$$y_1 - y_3 = \frac{1}{66}$$

$$y_3 = y_1 - \frac{1}{66} \ldots\ldots(4)$$

$$\textit{From } (1)\quad y_2 = \frac{1}{64}\left[1 + 132y_1\right] = \frac{1}{64} + \frac{132}{64}y_1 \ldots\ldots\ldots\ldots(5)$$

Substitute (4), (5) in (2)

https://doi.org/10.5281/zenodo.15289913

$$32y_1 - 66\left[\frac{1}{64} + \frac{132}{64}y_1\right] + 32\left[y_1 - \frac{1}{66}\right] = 1$$

$$32y_1 - \frac{66}{64} - \frac{8712}{64}y_1 + 32y_1 - \frac{32}{66} = 1$$

$$-72.125y_1 = 2.516$$

$$y_1 = -0.03488$$

From (4), $y_3 = y_1 - \frac{1}{66} = -0.03488 - \frac{1}{66} = -0.5$

From (5), $y_2 = \frac{1}{64} + \frac{132}{64}(-0.03488) = -0.0563$

6. Solve the following by finite difference method $y'' - y = 0$ given $y(0) = 0,\ y(1) = 1$ with $h = 0.25$.

QC 80610 MA 6452 NOV 2016

Given $h = 0.25$, $n = 4$	
x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.25$	$y_1 = y(0.25) = ?$
$x_2 = 0.5$	$y_2 = y(0.5) = ?$
$x_3 = 0.75$	$y_3 = y(0.75) = ?$
$x_4 = 1$	$y_4 = y(1) = 1$

Given differential equation is $\frac{d^2y}{dx^2} = y$

The finite difference scheme for the given differential equation is

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = y_i,\ i = 1,\ 2,\ 3$$

$$16\left(y_{i+1} + y_{i-1} - 2y_i\right) = y_i$$

$$16y_{i+1} - 33y_i + 16y_{i-1} = 0$$

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$16y_2 - 33y_1 + 16y_0 = 0$

$16y_3 - 33y_2 + 16y_1 = 0$

$16y_4 - 33y_3 + 16y_2 = 0$

Substituting the initial conditions, we have

$16y_2 - 33y_1 = 0$....................(1)

$16y_3 - 33y_2 + 16y_1 = 0$.........(2) i.e.

$-33y_3 + 16y_2 = -16$............(3)

(1) – (3) gives

$$-33y_1 + 33y_3 = 16$$

$$-y_1 + y_3 = \frac{16}{33}$$

$$y_3 = \frac{16}{33} + y_1 \text{......(4)}$$

$$\textit{From } (1) \quad y_2 = \frac{33}{16}y_1 \text{............(5)}$$

Substitute (4), (5) in (2)

$$16y_3 - 33y_2 + 16y_1 = 0$$

$$16\left[\frac{16}{33} + y_1\right] - 33\left[\frac{33}{16}y_1\right] + 16y_1 = 0$$

$$\frac{256}{33} + 16y_1 - \frac{1089}{16}y_1 + 16y_1 = 0$$

$$-36.0625y_1 = -7.7575$$

$$y_1 = 0.2151$$

From (4), $y_3 = y_1 + \frac{16}{33} = 0.2151 + \frac{16}{33} = 0.6999$

From (5), $y_2 = \frac{33}{16}(0.2151) = 0.4436$

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7. **Solve the boundary value problem $y'' = xy$ subject to the conditions $y(0) + y'(0) = 1,\ y(1) = 1$, taking $h = \frac{1}{3}$, by finite difference method.** **QC 41316 MA 6459 MAY 2018**

Given $h = \frac{1}{3}$, $n = 3$	
x	y
$x_0 = 0$	$y_0 = y(0) = ?$
$x_1 = \frac{1}{3}$	$y_1 = y\left(\frac{1}{3}\right) = ?$
$x_2 = \frac{2}{3}$	$y_2 = y\left(\frac{2}{3}\right) = ?$
$x_3 = 1$	$y_3 = y(1) = 1$

The finite difference form of first boundary condition is $y_0 + \frac{y_1 - y_{-1}}{2h} = 1$, since $y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$

$$2y_0 + 3(y_1 - y_{-1}) = 2$$

$$y_{-1} = \frac{2y_0 + 3y_1 - 2}{3} \text{......(1)}$$

The second boundary condition is $y_3 = 1$.......(2)

Given differential equation is $\frac{d^2y}{dx^2} = xy$. The finite difference scheme for the given differential equation is $\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = x_i y_i,\ i = 0, 1, 2$

$$9(y_{i+1} + y_{i-1} - 2y_i) = x_i y_i$$

$$9y_{i-1} - 18y_i - x_i y_i + 9y_{i+1} = 0$$

$$y_{i-1} - \left(2 + \frac{1}{9}x_i\right)y_i + y_{i+1} = 0$$

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Put $i = 0, 1, 2,$ we get

$$y_{-1} - \left(2 + \frac{1}{9}x_0\right)y_0 + y_1 = 0 \ldots\ldots.(3)$$

$$y_0 - \left(2 + \frac{1}{9}x_1\right)y_1 + y_2 = 0 \ldots\ldots..(4)$$

$$y_1 - \left(2 + \frac{1}{9}x_2\right)y_2 + y_3 = 0 \ldots\ldots..(5)$$

Substituting the boundary conditions (1), (2) in (3), (4), (5) we have,

$$\frac{2y_0 + 3y_1 - 2}{3} - \left(2 + \frac{1}{9}(0)\right)y_0 + y_1 = 0$$

$$2y_0 + 3y_1 - 2 - 6y_0 + 3y_1 = 0$$

$$-4y_0 + 6y_1 = 2$$

$$-2y_0 + 3y_1 = 1 \ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

$$y_0 - \left(2 + \frac{1}{9}.\frac{1}{3}\right)y_1 + y_2 = 0$$

$$y_0 - \frac{55}{27}y_1 + y_2 = 0 \ldots\ldots\ldots\ldots(7)$$

$$y_1 - \left(2 + \frac{1}{9}\frac{2}{3}\right)y_2 + 1 = 0$$

$$y_1 - \frac{56}{27}y_2 = -1 \ldots\ldots\ldots\ldots\ldots..(8)$$

Solving (6), (7), (8), by Gauss elimination method, we have

$$(A, B) = \begin{pmatrix} \boxed{1} & -\frac{55}{27} & 1 & 0 \\ -2 & 3 & 0 & 1 \\ 0 & 1 & -\frac{56}{27} & -1 \end{pmatrix}$$

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$$\approx \begin{pmatrix} 1 & -\frac{55}{27} & 1 & 0 \\ 0 & \boxed{-\frac{29}{27}} & 2 & 1 \\ 0 & 1 & -\frac{56}{27} & -1 \end{pmatrix} R_2 \to R_2 + 2R1$$

$$\approx \begin{pmatrix} 1 & -\frac{55}{27} & 1 & 0 \\ 0 & -\frac{29}{27} & 2 & 1 \\ 0 & 0 & -\frac{166}{729} & -\frac{2}{27} \end{pmatrix}$$

By back substitution,

$$-\frac{166}{729} y_2 = -\frac{2}{27} \qquad -\frac{29}{27} y_1 + 2y_2 = 1 \qquad y_0 - \frac{55}{27} y_1 + y_2 = 0$$

$$y_2 = \frac{2}{27} \times \frac{729}{166} = 0.3253 \qquad -\frac{29}{27} y_1 + 2(0.3253) = 1 \qquad y_0 - \frac{55}{27}(-0.3253) + 0.3253 = 0$$

$$y_1 = -0.3253 \qquad y_0 = -0.9879$$

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EXERCISE

Taylor Method

1 Given $\frac{dy}{dx} = 1 + y^2$, where $y = 0$ when $x = 0$, find $y(0.2)$, $y(0.4)$ and $y(0.6)$, using Taylor series method. (QC E3126 MA 2266 MAY 2010)

2 Using Taylor method, compute $y(0.2)$ and $y(0.4)$ correct to 4 decimal places given $\frac{dy}{dx} = 1 - 2xy$ and $y(0) = 0$, by taking $h = 0.2$. (QC 11395 MA 2266 MAY 2011)

Euler Method

2 Solve by Euler's method, the equation $\frac{dy}{dx} = (x + y)$, $y(0) = 0$,, choose $h = 0.2$ and compute $y(0.4)$ and $y(0.6)$. (QC 31528 MA 2266 NOV 2013)

Modified Euler Method

1 Compute y at $x = 0.1,\ 0.2$ given that $\frac{dy}{dx} = (x + y^2)$, $y(0) = 1$, using modified Euler's method.

(QC 90346 MA 8491 NOV 2019)

2 Evaluate $y(1.2)$ and $y(1.4)$ correct to three decimal places by the modified Euler method, given that $\frac{dy}{dx} = (y - x^2)^3$; $y(1) = 0$ taking $h = 0.2$. (QC 51579 MA 2266 MAY 2014)

Runge Kutta Method

1 Use Runge Kutta method of the fourth order to find $y(0.2)$, given that $y\frac{dy}{dx} = y^2 - x$, $y(0) = 2$.

(QC 53252 MA 6459 MAY 2019)

Milne's Predictor Corrector Method

1 Using Milne's predictor corrector method, find $y(0.4)$, given that $\frac{dy}{dx} = \frac{y^2}{2}(1 + x^2)$, $y(0) = 1$, .

$y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. (QC 31528 MA 2266 NOV 2013)

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2 Compute $y(0.5)$, $y(1)$ and $y(1.5)$ using Taylor's series for $\frac{dy}{dx}=\frac{1}{2}(x+y)$ with $y(0)=2$ and hence find $y(2)$ using Milne's method. (QC 27331 MA 6452 NOV 2015)

Adam Bashforth Predictor Corrector Method

2 Given $\frac{dy}{dx}=x^2(1+y)$ and $y(1)=1$, $y(1.1)=1.233$, $y(1.2)=1.548$, $y(1.3)=1.974$, evaluate $y(1.4)$ by Adam Bashforth predictor corrector method. (QC E3126 MA 2266 MAY 2010)

Finite Difference Method

1 Using finite difference method solve the boundary value problem $y''+3y'-2y=2x+3$, $y(0)=1$, $y(1)=1$ with $h=0.2$. (QC E3126 MA 2266 MAY 2010)

1 Solve the BVP $y''+y=0$, $y(0)=1$, $y(1)=0$ using finite difference method, taking $h=0.25$. (QC 51579 MA 2266 MAY 2014)

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